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TABLE OF CONTENTS

LOGIC AND FOUNDATIONS . . . . . p. 119
SET THEORY 120
COMBINATORIAL ANALYSIS 121
ORDER, LATTICES 121
GENERAL MATHEMATICAL SYSTEMS 122

THEORY OF NUMBERS 122
FIELDS 124
LINEAR ALGEBRA 124
ASSOCIATIVE RINGS AND ALGEBRAS 125
NON-ASSOCIATIVE RINGS AND ALGEBRAS 125

GROUPS AND GENERALIZATIONS 126
TOPOLOGICAL GROUPS AND LIE THEORY 126

FUNCTIONS OF REAL VARIABLES 129
MEASURE AND INTEGRATION 130
FUNCTIONS OF A COMPLEX VARIABLE 131
FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS 133

SPECIAL FUNCTIONS 135
ORDINARY DIFFERENTIAL EQUATIONS 136
PARTIAL DIFFERENTIAL EQUATIONS 142
POTENTIAL THEORY 145
FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS 146

SEQUENCES, SERIES, SUMMABILITY 147
APPROXIMATIONS AND EXPANSIONS 147
FOURIER ANALYSIS 149
INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS 149
INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS 149
FUNCTIONAL ANALYSIS 150
CALCULUS OF VARIATIONS 152

GEOMETRY . . . . . p. 153
CONVEX SETS AND GEOMETRIC INEQUALITIES 153

GENERAL TOPOLOGY, POINT SET THEORY 153
ALGEBRAIC TOPOLOGY 156
DIFFERENTIAL GEOMETRY, MANIFOLDS 158

PROBABILITY 173
STATISTICS 180

NUMERICAL METHODS 184
COMPUTING MACHINES 189

MECHANICS OF PARTICLES AND SYSTEMS 189
STATISTICAL THERMODYNAMICS AND MECHANICS 191
ELASTICITY, PLASTICITY 193
STRUCTURE OF MATTER 200
FLUID MECHANICS, ACOUSTICS 200
OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS 210
CLASSICAL THERMODYNAMICS, HEAT TRANSFER 213

QUANTUM MECHANICS 214
RELATIVITY 220
ASTRONOMY 222
GEOPHYSICS 225

OPERATIONS RESEARCH, ECONOMETRICS, GAMES 226
BIOLOGY AND SOCIOLOGY 228
INFORMATION AND COMMUNICATION THEORY 229
SERVOMECHANISMS AND CONTROL 229

HISTORY AND BIOGRAPHY 230
MISCELLANEOUS 231

AUTHOR INDEX

Adler, A. 902
Agekian, T. A. 1208
Agostinelli, C. 1147
Agranovich, V. M. 1126
Ahn, Jae-koo 888
Alvaryan, S. A. 942
Akhiezer, A. I. 1117
Albert, A. A. 681
Aleksandrov, A. D. 906, 907, 908, 909
Alexiewicz, A. 802
Alfsen, E. M. 709
Alfvén, H. 1203
Aljančić, S. 789
Allen, F. J. 1038
Allendoerfer, C. B. 868
Amati, G. 1171
Anderson, J. L. 1195
Andreoletti, J. 1220
Archbold, J. W. 960
Arhangel'skiĭ, Yu. A. 990
Artobolevskiy, I. I. 986
Arżanyh, I. S. 772, 1060
Balada, E. 738
Bakel'man, I. Ya. 905
Baker, W. E. 1038
Balabanov, V. A. 825
Balakrishnan, A. V. 775
Balaz, N. L. 998
Baltrukonis, J. H. 1057
Banaschewski, B. 807
Bantegui, C. G. 946
Bar-Hillel, Y. 648
Barnett, M. P. 1110
Barron, D. W. 1127
Balsarin, G. P. 931
Bass, R. W. 1214
Bassali, W. A. 1017
Bat', G. A. 1000
Baum, F. A. 1210
Beale, E. M. L. 1225
Bearman, R. J. 999
Beck, A. 697
Bel, L. 1194
Bellman, R. 828, 1228
Belousov, S. L. 1218
Bendixson, I. 752
Bennett, W. R. 932
Berezin, O. A. 1098
Berezina, L. Ya. 843
Berger, M. J. 1128
Bergström, H. 917
Bernstein, I. B. 1122
Bhagavantam, S. 1004
Bhatnagar, P. L. 1114
Bielynicki-Birula, A. 650
Bilodeau, G. 1104
Bilenkij, S. M. 1155
Biot, M. A. 1112
Bittner, R. 818
Bland, D. R. 1046
Blin-Stoyle, R. J. 1176
de Boer, J. H. 673
Bogolyubov, N. N. 1150
Boguz, W. 1035
Bohm, D. 1124, 1125
Borg, C. L. 1030
Borsuk, K. 858
Bose, R. C. 951
Box, G. E. P. 949
Brainerd, B. 806, 806
Braithwaite, R. B. 1237
Brauer, H. 879
Brebner, G. G. 1107
Breitenberger, E. 1129
Bremermann, H. J. 727
Brown, S. C. 1123
Bruhat, F. 889
Bryant, P. R. 1134
Buck, R. C. 713
Budden, K. G. 1127
Busemann, H. 913
Caianiello, E. R. 1161
Callaway, J. 1059
Călugăreanu, G. 706
Caregradskiĭ, I. P. 916
Carnahan, P. D. M. 968
Castellano, V. 938
Cattabriga, L. 767
Cattaneo, C. 1201
Chadan, K. 1151
Chakraborty, S. K. 1042
Chang, Ou-Yang 717
Chattarji, P. P. 1023
Chillingworth, H. R. 814a–b
Choquet, G. 778
Chow, Tse-Sun 763
Chung, K. L. 922
Čistyakov, V. P. 926
Clarion, C. 1080, 1081
Clinton, W. L. 1168
Collins, W. D. 1011
Comét, S. 692
Corcos, G. M. 1089
Cornea, A. 720
Corson, H. H. 697
Cotlar, M. 809
Coxeter, H. S. M. 841, 848
Craig, H. V. 886
Cristescu, R. 710
Cuesta Dutari, N. 651
Cugiani, M. 715
Cureton, E. Z. 1242
Curtiss, C. F. 1108, 1109, 1110
Danos, I. 668
Darevskiĭ, V. M. 1056
Darlington, S. 1243
Datta Majumdar, S. 1169, 1183
Daubin, S. C. 1229
David, H. A. 1238
Davies, D. R. 1088
Deisage, H. 662
Dem'ianov, I. A. 1016
Denjoy, A. 739a–c
Derebewka, H. 1053
Destouches, J.-L. 1181
Devingtal', Yu. V. 769
Diendonné, J. 702
Dieulefait, C. E. 969
Dijksterhuis, E. J. 1258
Dmitrieva, M. I. 1127
Dobrescu, A. 700, 701
Doležal, V. 745
Drahos, I. 988
Drukarev, G. F. 1182
Duda, R. 857
Dvořák, M. 1236
Eells, J., Jr. 868
Ehrlich, L. W. 957
Eidel'man, S. D. 771
Eisenberg, E. 914
Elder, J. W. 1035
Elmaghraby, S. E. 1233
Endl, K. 777
Engel'son, Ya. L. 826
Epstein, S. T. 903
Erdös, P. 853, 812, 876
Erohin, V. 779
Eum, Sang-Seup 895
Evett, A. A. 1135
Evgrafov, M. A. 798
Fain, V. M. 1148
Fedorov, F. I. 1179
Feller, W. 744, 770
Ferrari, E. 1160
Filin, A. P. 962
de Finetti, B. 924
Fisher, I. Z. 1197
Fisher, R. A. 935
Fixman, M. 999
Florian, A. 846
Foias, C. 822
Foldy, L. L. 1145
Fox, D. W. 762
Fox, L. 972
Fraenkel, A. A. 648
Fraeys de Veubeke, B. 994
Francis, G. 783
Fridlender, V. R. 774
Friedman, A. 765, 766, 1138
Fröliober, A. 890
Fujikawa, H. 1084

(*Continued on cover 3*)

Journal references in Mathematical Reviews are now given in the following form: J. Brobdingnag. Acad. Sci. (7) 4 (82) (1952/53), no. 3, 17–42 (1954), where after the abbreviated title one has: (series number) volume number (volume number in first series if given) (nominal date), issue number if necessary, first page–last page (imprint date). In case only one date is given, this will usually be interpreted as the nominal date and printed immediately after the volume number (this a change from past practice in Mathematical Reviews where a single date has been interpreted as the imprint date). If no volume number is given, the year will be used in its place.

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LOGIC AND FOUNDATIONS

See also 915.

645:

★**Lorenzen, Paul. Formale Logik.** Sammlung Göschen Bd. 1176–1176a. Walter de Gruyter and Co., Berlin, 1958. 165 pp. DM 4.80.

The content of this compact book is accurately described by the author's summary: "Die Darstellung folgt im grossen der historischen Entwicklung: (1) aristotelische Syllogistik, (2) Aussagenlogik, (3) Prädikatenlogik, (4) Prädikatenlogik mit Identität. Die Syllogistik wird mit einer neuen relationentheoretischen Methode behandelt, die Aussagenlogik zunächst axiomatisch (mit 'Implikation' als Grundbegriff wie in der Verbandstheorie), dann als Kalkül. Hier ist ein eigenes Kapitel der effektiven und ihrer operativen Deutung gewidmet. Für die Prädikatenlogik (mit und ohne Gleichheit) wird der Gödel'sche Vollständigkeitssatz bewiesen unter Benutzung einer Vereinfachung der Gentzenkalküle. Aus der Existenz unentscheidbarer Kalküle (Beweis vereinfacht nach Post in §9) wird der Church'sche Unentscheidbarkeitssatz und ein allgemeiner Gödel'scher Unvollständigkeitssatz gefolgert. Für die Logik mit Gleichheit werden die ι-Terme und ihre Elimination behandelt, ferner die Einführung abstrakter Objekte, speziell der Relationen und Funktionen als 'façon de parler'. Das Buch schliesst mit der Erweiterbarkeit unendlicher Modelle gleichheitslogischer Axiomensysteme."

The text is enriched with historical remarks as well as references to contemporary literature. The author's point of view *Einführung in die operative Logik und Mathematik*, Springer-Verlag, Berlin–Göttingen–Heidelberg, 1955; MR **17**, 223] provides novelty at points which adds to the interest of the book, although it makes the book less suitable as an introduction to formal logic than as supplementary reading for students of logic.

P. C. Gilmore (Yorktown Heights, N.Y.)

646:

Székely, David L. Formalisierung der Einheitswissenschaft mit Dimensionsbegriffen von variabler Exaktheit. Synthese **10**, 148–170.

Bei der oben zitierten Arbeit handelt es sich um einen skizzenhaften Bericht über einen Versuch, eine formalisierte Einheitssprache für möglichst viele Einzelwissenschaften zu konstituieren. Diese Einheitssprache ist als eine Art universeller Metasprache für die Sprachen der Einzelwissenschaften gedacht. In der Einheitssprache sollen u.a. die Beziehungen der einzelnen Wissenschaftssprachen untereinander, und ihre Exaktheitsarten und Exaktheitsgrade erfassbar sein. Dabei bleibt z.B. unklar, wann eine Wissenschaftssprache etwa von der Exaktheitsart "mittel-exakt" ist. Die angegebenen Formalisierungen der Einheitssprache lassen keinen Vergleich mit den gewohnten Formalisierungen mathematischer und physikalischer Theorien zu. Es muss gesagt werden, dass zumindest die vorliegende sehr summarische Arbeit kein genaues Bild von dem Versuch gibt und deswegen auch kein abschliessendes Urteil gestattet. Nach Ansicht des Ref. ist jedoch ein derart breit angelegtes Unterfangen zumindest heute noch sehr zweifelhaft. Man denke doch nur daran, welch grosse Schwierigkeiten sich schon bei einem adäquaten Aufbau einer Sprache der sog. exakten Wissenschaften ergeben.

H. Kiesow (Münster)

647:

Lyubčenko, G. G. Methods for determining the identical truth or falsity of the calculus of assertion formulae of bivalent logic. Dopovidi Akad. Nauk Ukrain. RSR **1958**, 1153–1156. (Ukrainian. Russian and English summaries)

The author gives five sets of rules for determination of identically true statements of the propositional calculus. (The rules have been formulated for application by computing machines.) In effect, the rules reduce a given proposition to a sort of canonical form which may then be compared to the members of the set corresponding to the identically true propositions. The procedure differs from current ones in that (among other aspects) each rule is a Markov substitution [A. A. Markov, *The theory of algorithms*, Trudy Mat. Inst. Steklov, no. 42, Izdat. Akad. Nauk SSSR, Moscow, 1954; MR **17**, 1038], the procedure is not bound to application of the distributive law, and reduction to normal form is not necessary.

R. M. Baer (Berkeley, Calif).

648:

★**Fraenkel, Abraham A.; and Bar-Hillel, Yehoshua. Foundations of set theory.** Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1958. x+415 pp. 42 guilders; $11.00.

Ein wirklich gutes Buch, mit grosser Sachkenntnis geschrieben, mit einem besonders verdienstvollen und aufschlussreichen Kapitel "Metamathematical and semantical approaches" (Kapitel V). Es bringt Zusammenfassungen der verschiedenen wichtigen und teilweisen Theorien, die bis 1957 (einschliesslich) zu dem Thema "Grundlagen der Mengenlehre" entstanden und veröffentlicht wurden. Es kann zugleich als Lehrbuch und besonders als Nachschlagewerk dienen. Bei der Anzahl der Arbeiten, die auf diesem Gebiet erschienen sind (die Bibliographie am Ende des Buches, die nicht alle diese Arbeiten enthält, ist 50 Seiten stark) ist es verständlich, dass nicht alle Theorien und Halbtheorien gleichmässig voll besprochen werden konnten, zumal der Umfang des Buches nicht allzu gross werden sollte. Aber im ganzen kann man wohl sagen, dass die Verfasser ihr bestes getan haben, um eine möglichst objektive Darstellung der

verschiedenen Gegenstände zu bringen. Dabei haben sie versucht, die verschiedenen Theorien und Teiltheorien, trotzdem sie vielfach unabhängig voneinander entstanden sind, einander zu nähern, wenn auch eine nur annähernde Synthese zur Zeit nicht möglich ist.

Kapitel I enthält eine eher kurze Übersicht über die logischen und semantischen Antinomien, die seinerzeit die Forschungen auf dem Gebiet der Grundlagen der Mathematik und speziell der Mengenlehre überhaupt eingeleitet haben. Das zweite Kapitel enthält eine Übersicht über die Axiomensysteme von Zermelo-Fraenkel, J. von Neumann, P. Bernays und K. Gödel. Das Auswahlaxiom findet in diesem Rahmen eine breite und ausführliche Darstellung. Kapitel III ist der Typentheorie Russells und Whiteheads und verwandten Richtungen gewidmet. Kapitel IV ist eine Darstellung des Intuitionismus, die in der breiten Ausführlichkeit, die Professor Fraenkel überhaupt zueigen ist, gehalten ist. Kapitel V schliesslich, welches das bedeutendste des ganzen Buches ist, ist den metamathematischen und semantischen Richtungen gewidmet. Dabei werden u.a. die negativen Theoreme von Gödel, Taraki, Church und Church-Rosser besprochen und wahre Horden mathematisch-logischer Begriffe, die meistens in letzter Zeit entstanden sind, eingeführt. Zum Schluss dieses Kapitels folgen philosophische Bemerkungen über den realistischen, nominalistischen, konzeptualistischen und antimetaphysischen Standpunkt zur Universalienfrage in der Mathematik.

Wir möchten auf einen Schönheitsfehler hinweisen. Auf Seite 341 steht "asserting $F(x)$ is to assert that all x are F". Aber $F(x)$ kann, wenn wir uns nicht irren, nicht behauptet werden und noch weniger äquivalent zu $(\forall x)F(x)$ sein, weil es eine Urteilsform und kein Urteil ist [vgl. jedoch Church, *Introduction to mathematical logic*, vol. I, Princeton Univ. Press, Princeton, N.J., 1956; MR **18**, 631; p. 46].

Es ist wünschenswert, dass das Register am Schluss des Werkes ausführlicher gehalten wird, damit das Buch wirklich als Nachschlagewerk dienen kann.

B. Germansky (Berlin)

649:

Guillaume, Marcel. Rapports entre calculs propositionnels modaux et topologie impliqués par certaines extensions de la méthode des tableaux sémantiques. Système S4 de Lewis. C. R. Acad. Sci. Paris **246** (1958), 2207–2210.

The author's abstract: "Après la définition—parallèle à celle de la Note sur le système de Feys-von Wright—de tableaux sémantiques pour le système S4 [C. I. Lewis, *A survey of symbolic logic*, Univ of California Press, Berkeley, 1918], un raisonnement analogue conduit à de nouvelles précisions sur les rapports entre le système S4 et certaines familles d'espaces topologiques."

B. A. Galler (Ann Arbor, Mich.)

650:

Białynicki-Birula, A.; and Rasiowa, H. On constructible falsity in the constructive logic with strong negation. Colloq. Math. **6** (1958), 287–310.

This article is a continuation of a paper of Rasiowa [Fund. Math. **46** (1958), 61–80; MR **20** #5137] in which an algebraic characterization of the system $\mathscr{S}$ of the constructive propositional calculus with strong negation was given. The terminology and the notation is here the same as in that paper. The knowledge of that paper is assumed here.

The idea of the above-mentioned constructive logic with strong negation is due to David Nelson [J. Symb. Logic **14** (1949), 16–26; MR **10**, 669], who introduced a constructive interpretation for logical connectives of the number theory and characterized a formal system of the number theory satisfying this interpretation. An analogical system of the number theory was later investigated by Markov [Uspehi Mat. Nauk (N.S.) **5** (1950), no. 3 (37), 187–188]. The system of the propositional calculus with strong negation was examined by N. N. Vorob'ev [Dokl. Akad. Nauk SSSR. **85** (1952), 456–468, 689–692; MR **14**, 233].

Under Nelson's interpretation of logical connectives the strong negation of a conjunction $\sim(\alpha\cdot\beta)$ is valid in case at least one of the formulas $\sim\alpha$, $\sim\beta$ is valid and a formula $\sim\prod_{x_k}\alpha(x_k)$ is valid if and only if there exists such an x_p that $\sim\alpha\binom{x_p}{x_k}$ is valid.

This paper deals with the above-mentioned system $\mathscr{S}$ and with the system $\mathscr{S}^*$ of the functional calculus based on $\mathscr{S}$. The algebraic characterization of $\mathscr{S}$ is generalized to $\mathscr{S}^*$. Using algebraic and topological methods the authors prove that according to the idea of Nelson a formula $\sim(\alpha\cdot\beta)$ is provable in $\mathscr{S}$ or in $\mathscr{S}^*$ if and only if at least one of the formulas $\sim\alpha$, $\sim\beta$ is provable. Similarly, a formula $\sim\prod_{x_k}\alpha(x_k)$ is provable in $\mathscr{S}^*$ if and only if for some x_p the formula $\sim\left(\alpha\binom{x_p}{x_k}\right)$ is provable. The theorems mentioned above are equivalent to the theorems stating that the disjunction $\alpha+\beta$ is provable in $\mathscr{S}$ or in $\mathscr{S}^*$ if and only if at least one of the formulas α, β is provable and that a formula $\sum_{x_k}\alpha(x_k)$ is provable if and only if for some x_p the formula $\alpha\binom{x_p}{x_k}$ is provable. The decidability of formulas of $\mathscr{S}^*$ having the prenex normal form follows from the last theorem.

Moreover, the algebraic method of examination of the systems $\mathscr{S}$ and $\mathscr{S}^*$ permits the authors to show other theorems concerning provability in these systems. E.g., a formula without the sign of strong negation is provable in $\mathscr{S}$ or in $\mathscr{S}^*$ if and only if it is provable in the Heyting propositional calculus or in the Heyting functional calculus, respectively, a result first stated by Vorob'ev.

Some other connections between provability in $\mathscr{S}$ and in the Heyting propositional calculus lead to the possibility of constructing simple functionally free $\mathscr{N}$-lattices of sets; e.g., it is proved that every $\mathscr{N}$-lattice connected with the Heyting algebra of all open subsets of an n-dimensional Euclidean space is functionally free.

A very simple decision method is given for formulas of $\mathscr{S}$ having the form $\alpha\equiv\beta$ (strong equivalence) where α and β are formulas without the sign of implication and without the sign of intuitionistic negation.

B. A. Galler (Ann Arbor, Mich.)

SET THEORY

651:

Cuesta Dutari, Norberto. Mathematics of order. I, II. Rev. Acad. Ci. Madrid **52** (1958), 147–321. (Spanish)

We have here the first two chapters of a four-chapter treatise on ordered sets from the standpoint of naive set

theory. The first chapter deals with the notions of set and power of a set, the second with well-ordered sets, ordinal numbers, and alephs. There are exercises, harder problems, and bibliographies. The third and fourth chapters are to be devoted to totally and partially ordered sets, respectively. *F. Bagemihl* (Notre Dame, Ind.)

652:

Erdös, P.; and Rado, R. Partition relations connected with the chromatic number of graphs. J. London Math. Soc. **34** (1959), 63–72.

The chromatic number of a graph Γ is defined, in analogy with the usual definition of the chromatic number of a finite graph, as the least cardinal number a having the property that the nodes of Γ can be divided into a set of a mutually disjoint classes in such a way that no two nodes in the same class are joined by an edge. In the first part of this paper a construction is given for a graph without triangles having chromatic number a, where a is any cardinal $\geqq \aleph_0$. Assuming a certain form of the general continuum hypothesis it is further shown that a graph with these properties can be constructed whose set of nodes has cardinal a.

In the second part of the paper the following new partition relation is introduced: If A is a set, b is a cardinal number and Λ is a set of sets, then $A \to (b, \Lambda)^2$ expresses the fact that if $[A]^2 = K_0 + K_1$, then there is a subset X of A such that either $[X]^2 \subset K_0$ and $|X| = b$ or $[X]^2 \subset K_1$ and $X \in \Lambda$. If Ω is a set of sets, then the set A is said to be of first Ω-category if there is an $\Omega' \subset \Omega$ such that $|\Omega'| < |\Omega|$ and $A \subset \sum (X \in \Omega') X$; otherwise A is said to be of second Ω-category. In this terminology the following theorem is proved. Suppose Ω is a set of sets and $|\Omega|$ is a regular infinite cardinal. Let A be a set of second Ω-category and let Λ_2 denote the set of all subsets of A which are of second Ω-category. Then $A \to (\aleph_0, \Lambda_2)^2$. It is further shown with the help of the first result of the first part of the paper, concerning the existence of triangle-free graphs with chromatic number a, that corresponding to any infinite singular cardinal a there is an Ω with $|\Omega| = a$ for which the theorem does not hold, provided a form of the general continuum hypothesis is assumed to be true.

G. A. Dirac (Hamburg)

COMBINATORIAL ANALYSIS

See also 652, 840, 841, 950, 951, 1230.

653:

Harary, Frank; and Prins, Geert. The number of homeomorphically irreducible trees, and other species. Acta Math. **101** (1959), 141–162.

The species of trees enumerated in this paper are of two kinds, those obtained by considering special properties specifying a given tree, and those obtained by weakening the definition of a tree. Examples of the first are trees of given diameter and trees of given "partition", specified by a product $t_1^{a_1} t_2^{a_2} \cdots$, with a_i the number of points of degree i. Examples of the second are trees of given strength; a tree has strength s if at least one pair of points is joined by s lines, but no two points are joined by more than s lines. Enumerations are given for "planted" (rooted at an endpoint) and rooted trees as well as for unrooted (free) trees, and, for the most part, both points (other than the root) and lines are regarded as alike. As usual the enumeration of trees is given by a formula, obtained from Otter's theorem [Ann. of Math. (2) **49** (1948), 583–599; MR **10**, 53], relating their enumerating generating function to the generating function either of rooted trees or planted trees (of the same species) or both. The homeomorphically irreducible trees of the title are trees with no point of degree two; this characterization bifurcates the whole class. Another species enumerated, that of trees whose automorphism group is the identity, is also bifurcative. Also enumerated are weighted trees (each point is assigned a positive integer as its weight and the weight of the tree is the sum of weights of its points), and oriented, directed and signed trees, these adjectives referring to conditions on the lines. All enumerating formulas are given in a compact symbolic form and no attention is paid to actual computation of the numbers.

Three appendices give diagrams of all trees with at most 10 points, of all homeomorphically irreducible trees with at most 12 points, and of all trees whose automorphism group is the identity with at most 12 points.

J. Riordan (New York, N.Y.)

ORDER, LATTICES

See also 709, 1230.

654:

Guillaume, Marcel. Calculs de conséquences et tableaux d'épreuve pour les classes algébriques générales d'anneaux booléiens à opérateurs. C. R. Acad. Sci. Paris **247** (1958), 1542–1544.

For subsets P and Q of the set of Boolean functions on a Boolean algebra with operators, the author defines "exactness" of $P \vdash Q$. He then exhibits and discusses the implications of a set of rules for determining whether $P \vdash Q$ is or is not "exact". (Terminology and concepts are related to those in the paper reviewed as #649.)

B. A. Galler (Ann Arbor, Mich.)

GENERAL MATHEMATICAL SYSTEMS

655:

Ingleton, A. W. A note on independence functions and rank. J. London Math. Soc. **34** (1959), 49–56.

A linear independence of vectors over a division ring is generalized to an abstract independence relation on a given set S. For this purpose "J-functions" are introduced which present a generalization of earlier considered "I-functions". If f is a J-function on S then $f(x_1, \cdots, x_n) = 1$ means that the elements x_i are "independent" and $f = 0$ that they are "dependent". A representation of (S, f) over a division ring is a mapping v of S into a vector space over a division ring such that $f(x_1, \cdots, x_n) = 1$ if and only if the vectors $vx_1, \cdots, vx_n$ are linearly independent.

R. Rado [Proc. London Math. Soc. (3) **7** (1957), 300–320; MR **19**, 522] proved a necessary condition for an I-function to be so representable. In the present note it is shown that

any J-function (and consequently any I-function) satisfies a somewhat stronger condition than Rado's one. It is also shown that the notion of "rank" of a subset [cf. H. Whitney, Amer. J. Math. **57** (1935), 509–533] can be defined relative to any J-function and that such rank functions are characterized (on finite subsets) by the increasing and subadditive properties. A simple example of an I-function on a set of 10 elements is given which is not representable over any division ring and an example of an I-function on a set of 9 elements is given which is representable over a division ring but not over any field.

M. Kolibiar (Bratislava)

THEORY OF NUMBERS

See also 791, 1230.

656:

Sanielevici, S. Une généralisation du théorème de Wilson. Com. Acad. R. P. Romîne **8** (1958), 737–744. (Romanian. Russian and French summaries)

Following the pattern of one of the classical proofs of Wilson's theorem, the author proves the following more general result: Given a positive integer n, set $P=\prod_{s>m,(s,m)=1} m$. Then $P\equiv -1 \pmod n$ if n has primitive roots; $P\equiv 1 \pmod n$ otherwise. The first congruence (of which Wilson's theorem is a particular case) is easily verified to hold in the four cases ($n=2, 4, p^s, 2p^s$) when primitive roots exist. In the proof of the second congruence, use is made of several case distinctions and of a basis for the (abelian, multiplicative) groups of residues coprime with n.

E. Grosswald (Princeton, N.J.)

657:

Riesel, Hans. Mersenne numbers. Math. Tables Aids Comput. **12** (1958), 207–213.

This paper gives a table, computed on the Swedish electronic computer BESK, of the smallest factors q of those Mersenne numbers $M_p=2^p-1$ for $p<10{,}000$, for which $q<10\cdot 2^{20}$. Known primes p for which M_p is prime, including 3217 which was tested on the BESK, are included in the table.

C. B. Haselgrove (Manchester)

658:

Malyšev, A. V. The relationship between the theory of the distribution of L-series zeroes and the arithmetic of ternary quadratic forms. Dokl. Akad. Nauk SSSR **122** (1958), 343–345. (Russian)

In an earlier paper [same Dokl. **118** (1958), 1078–1079; MR **21** #33] the author formulated a series of general theorems concerning the representation of a large integer m by positive ternary quadratic forms with odd prime invariants Ω, Δ. In certain of these theorems it was required that there exist an odd prime q such that $(-\Delta m/q)=1$. It is announced in the present paper that this condition can be dropped if the following weakening of the extended Riemann hypothesis holds: For sufficiently large m there is in the region $|s-1|<(\ln\ln m)^2 \ln\ln\ln m/\ln^{1/2} m$ no zero of the L-function $L(s)=\sum_{n=1}^{\infty}\chi(n)n^{-s}$, where $\chi(n)=(-4\Omega^2\Delta m/n)$ is a character (mod $4\Omega^2\Delta m$). The proof of one of the modified theorems is outlined.

W. J. LeVeque (Ann Arbor, Mich.)

659:

Tull, J. P. Dirichlet multiplication in lattice point problems. Duke Math. J. **26** (1959), 73–80.

The author establishes a theorem which he states is a generalization of a theorem on the convergence of the Dirichlet product given by Landau [E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Teubner, vol. II, Leipzig, 1909; pp. 755–758]. Theorem: $A(x)$ and $B(x)$ are of bounded variation on each interval $(1, x)$ with $A(1)=B(1)=0$: for $x\geqq 1$,

$$A(x)=\sum_{\mu=1}^{k} x^{\alpha_\mu}P_\mu(\log x)+O(x^{\alpha}\log^{l}(x+1)),$$

$$B(x)=\sum_{\nu=1}^{k} x^{\beta_\nu}Q_\nu(\log x)+O(x^{\beta}\log^{l'}(x+1)),$$

and the total variations of the main terms of the above are of order of magnitude

$$V_A(x)=O(x^{\rho}\log^{m}(x+1)),\quad V_B(x)=O(x^{\tau}\log^{m'}(x+1)),$$

respectively, where $\alpha, \beta, \rho, \tau, l, l', m, m'$ are non-negative real numbers; α_μ, β_ν are complex numbers; P_μ and Q_ν are polynomial functions. $C(x)$ is the Stieltjes resultant of A and B; i.e., $C(x)=\int_1^x A(x/u)dB(u)$. Then there is a function $T(x)$ of the same form as the main terms in $A(x)$ and $B(x)$ such that

$$C(x)=T(x)+O\{x^{\omega}\log^{q}(x+1)\},$$

$$\omega=(\rho\tau-\alpha\beta)/(\rho+\tau-\alpha-\beta),$$

$$q=\max\{r+r'+1,\ [(l+m')(\rho-\beta)+(l'+m)(\tau-\alpha)]/(\rho+\tau-\alpha-\beta)\}$$

if $\max(\alpha,\beta)<\min(\rho,\tau)$. If the last condition is not satisfied, other values for ω and q are obtained which are given in detail in the paper. Furthermore the total variation of $C(x)$ is given by $V_C(x)=O\{x^{\max(\rho,\tau)}\log^{m^*}(x+1)\}$, where $m^*=m$, m', or $m+m'+1$ according as $\rho>\tau$, $\rho<\tau$, or $\rho=\tau$, respectively.

D. Moskovitz (Pittsburgh, Pa.)

660:

Wirsing, Eduard. Eine Erweiterung des ersten Romanovschen Satzes. Arch. Math. **9** (1958), 407–409.

Let $f(x)$ be a polynomial with positive highest coefficient which assumes integral values for integer x. The author proves (extending previous results of Romanoff and Wirsing) that the density of integers of the form $p+f(q)$ (p, q primes) is positive.

P. Erdös (Boulder, Colo.)

661:

Palamà, Giuseppe. Su di una questione relativa alla partizione di n. Boll. Un. Mat. Ital. (3) **13** (1958), 558–563. (English summary)

Denoting by $(n, >m)$, the number of partitions of n into summands the least of which is greater than m, the author evaluates $(n, >[\frac{1}{3}n])$, thus extending some of the reviewer's results [see Proc. London Math. Soc. (2) **39** (1935), 142–149; J. London Math. Soc. **11** (1936), 278–280]. The case $(n, >[\frac{1}{3}n]-2)$ is also considered.

H. Gupta (Chandigarh)

662:

Delange, Hubert. On a theorem of Erdös. Scripta Math. **23** (1957), 109–116 (1958).

The author gives a new proof for the following theorem of the reviewer [Ann. of Math. (2) **47** (1946), 1–20; MR **7**, 416]. Let $f(n)$ be an additive function satisfying $f(p)\to 0$ as

$p \to \infty$ and $\sum_p (f^2(p)/p) = \infty$ (p prime). Then the number of integers $n \leq x$ satisfying $f(n) - [fn] \leq t$ equals $tx + o(x)$. Several further theorems on the number of prime factors of integers are proved by the author. His principal tools are Weyl's theorem on uniform distributions and Brun's method; probability theory is not used.

P. Erdős (Birmingham)

663:

Postnikov, A. G.; and Pyateckiĭ, I. I. On Bernoulli-normal sequences of symbols. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 501–514. (Russian)

A family of transformations T of a metric space R into itself is called a dynamical system, and a measure μ on R, with $\mu(R) = 1$, is said to be invariant if, for all measurable sets E, $T^{-1}E$ is measurable and $\mu T^{-1}E = \mu E$. We are thus concerned with measure-preserving set-transformations in the sense of J. L. Doob [*Stochastic processes*, Wiley, New York, 1953; MR **15**, 445; Chapter X]. Under these circumstances the author proves the ergodic theorem in the form that, for almost all $p \in R$,

$$\lim_{n \to \infty} n^{-1} \sum_{r=1}^{n} \varphi(T^{r-1}p) = \int_R \varphi(p) d\mu;$$

when φ is absolutely integrable and R cannot be decomposed into the union of two disjoint invariant sets of positive μ-measure.

This result is applied to the following situation. In what follows α_i is either 0 or 1; 1 corresponds to the occurrence of an event which has probability p and 0 corresponds to its non-occurrence, which has probability $q = 1 - p$. An infinite set of events or tests corresponds to a binary decimal $\alpha = \cdot\alpha_1\alpha_2\alpha_3\cdots$. A measure μ is defined on the space $R = [0, 1]$ ($0 \leq \alpha \leq 1$) as follows. If $A = \cdot\alpha_1\alpha_2\cdots\alpha_n$, then $\mu[A, A + 2^{-n}]$ is defined to be the probability of achieving the results $\alpha_1, \alpha_2, \cdots, \alpha_n$ in a succession of n tests. This measure is invariant for the set of transformations T^k ($k \geq 0$) defined by $T^k\alpha = \{\alpha 2^k\}$ (fractional part). If $\alpha = .\alpha_1\alpha_2\alpha_3\cdots$ and $\Delta = .\varepsilon_1\varepsilon_2\cdots\varepsilon_s$, then $N_P(\alpha, \Delta)$ denotes the number of times that Δ occurs as s successive digits in $\alpha_1\alpha_2\cdots\alpha_{P+s-1}$. The number α is said to be "normal according to Bernoulli" (because of the connexion with the Bernoulli or binomial distribution) if

$$\lim_{P \to \infty} \frac{N_P(\alpha, \Delta)}{P} = \mu\Delta$$

for all Δ and $s \geq 1$.

It is shown that the probability of a sequence of events being normal according to Bernoulli is 1; i.e., the μ-measure of the set of numbers α normal according to Bernoulli is 1. It is also shown that, if

$$\limsup_{P \to \infty} \frac{N_P(\alpha, \Delta)}{P} < C\mu\Delta,$$

for some constant C and all Δ and s, then α is normal according to Bernoulli.

An example based on the work of D. G. Champernowne [J. London Math. Soc. **8** (1933), 254–260] is given.

R. A. Rankin (Glasgow)

664:

Postnikov, A. G.; and Pyateckiĭ, I. I. A Markov-sequence of symbols and a normal continued fraction. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 729–746. (Russian)

The ideas of the paper reviewed above are extended and applied to a stationary Markov process, the results being of a similar type. Two kinds of dynamical system are considered. The first is similar to that introduced in the previous paper; i.e., the transformations are of the form $\{\alpha\} \to \{2\alpha\}$. In the second system the transformations are of the form $\{\alpha\} \to \{1/\alpha\}$ and involve the expression of α as a continued fraction. The associated measure μ is defined by

$$\mu\Delta = \frac{-1}{\log 2} \int_\Delta \frac{dx}{1+x}$$

on sets of numbers α having given binary digits $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_s$ in certain given places $i_1, i_2, \cdots, i_s$ ($s \geq 1$).

R. A. Rankin (Glasgow)

665:

Postnikov, A. G. A test for a completely uniformly distributed sequence. Dokl. Akad. Nauk SSSR **120** (1958), 973–975. (Russian)

Consider an infinite sequence $\{\alpha_i\}$ of real numbers with $0 \leq \alpha_i \leq 1$ for all i. Let I_s be an s-dimensional interval lying in the s-dimensional unit cube and let $N_k(I_s)$ be the number of points $(\alpha_i, \cdots, \alpha_{i+s-1})$, $1 \leq i \leq k$, contained in I_s. If there is a constant c such that

$$\limsup_{k \to \infty} N_k(I_s)/k < c|I_s|$$

for all $s \geq 1$ and all I_s, where $|I_s|$ is the volume of I_s, then $\{\alpha_i\}$ is completely uniformly distributed. This theorem and its proof are very similar to one concerning normal numbers published by the present author and I. I. Pjateckiĭ [see the article reviewed second above].

L. Schmetterer (Berkeley, Calif.)

666:

Postnikov, A. G. A criterion for testing the uniform distribution of an exponential function in the complex domain. Vestnik Leningrad. Univ. **12** (1957), no. 13, 81–88. (Russian. English summary)

A criterion for the uniform distribution of the exponential function on the real domain, which was given by the author and I. I. Pyateckiĭ [see the article reviewed third above] is extended to the complex domain. Let $N_P(\Delta)$ be the number of fractional parts $\{(\mu_1 + i\mu_2)(a + ib)^x\}$ ($0 \leq x < P$) lying in an arbitrary square Δ of area $|\Delta|$ in the interior of the unit square, the sides of Δ being parallel to the axes. Then, if there exists a constant C such that, for all Δ,

$$\limsup_{P \to \infty} \frac{N_P(\Delta)}{P} < C|\Delta|,$$

it is shown that the fractional parts are uniformly distributed in the unit square. Here $\{\mu_1 + i\mu_2\}$ is defined to be $\{\mu_1\} + i\{\mu_2\}$, and $a + ib$ is a Gaussian integer, which is not real and is different from $\pm i$. For given $a + ib$ an elaborate construction of a complex number $z = \mu_1 + i\mu_2$ is given, for which $\{z(a + ib)^x\}$ is shown to be uniformly distributed by applying the above criterion.

R. A. Rankin (Glasgow)

667:

Subba Rao, M. V. Some properties of quadratic residues. Math. Student **26** (1958), 7–8.

Two theorems on quadratic residues and non-residues, which are direct consequences of the reviewer's generalizations of Kelly's results, are proved using the algebra of quadratic residues. [See R. Vaidyanathaswamy, J. Indian Math. Soc. (N.S.) **2** (1937), 239–249.]

H. Gupta (Chandigarh)

668:

Dancs, I. On an extremal problem. Acta Math. Acad. Sci. Hungar. **9** (1958), 309–313.

A new proof is given of a result of E. Makai [same Acta **9** (1958), 105–110, MR **20** #1757]. As an application, an improvement is given of one of Turán's main theorems [*Eine neue Methode in der Analysis und deren Anwendungen*, Akadémiai Kiadó, Budapest, 1953; MR **15**, 688], which reads: If $z_1, \cdots, z_n$, $b_1, \cdots, b_n$ are complex numbers, and if m is an integer $\geqq -1$, then there exists an integer ν with $m+1 \leqq \nu \leqq m+n$ and

$$|b_1 z_1^{\nu} + \cdots + b_n z_n^{\nu}| \geqq |b_1 + \cdots + b_n| \left(\frac{n}{2e(m+n)}\right)^n \min_{1 \leqq j \leqq n} |z_j|^{\nu}.$$

The author replaces $(n/2e(m+n))^n$ by $(2e)^{-1}(n/2e(n+m))^{n-1}$.

N. G. de Bruijn (Amsterdam)

669:

Uchiyama, S. Sur les sommes de puissances des nombres complexes. Acta Math. Acad. Sci. Hungar. **9** (1958), 275–278.

If $z_1, \cdots, z_n$ are complex numbers, we put

$$K(z_1, \cdots, z_n) = \max_{1 \leqq \nu \leqq n} |z_1^{\nu} + \cdots + z_n^{\nu}|,$$

and let M_n denote the minimum of $K(z_1, \cdots, z_n)$ under the condition that $\max_{1 \leqq j \leqq n} |z_j| = 1$. It was shown by P. Turán [*Eine neue Methode in der Analysis und deren Anwendungen*, Akadémiai Kiadó, Budapest, 1953; MR **15**, 688] that $M_n > C_1 (\log n)^{-1}$ with a positive constant C_1, and this was improved by the reviewer [see the Chinese edition of Turán's book, Peking, 1956], who showed that $M_n > C_2 (\log \log n)/(\log n)$ with a positive constant C_2. The present author shows that $M_n \exp(M_n \sum_1^n \nu^{-1}) > 2 \cdot 3^{\frac{1}{2}}/\pi$, which implies that for all $\varepsilon > 0$ we have $M_n > (1-\varepsilon)(\log \log n)/(\log n)$ for all $n > n_0(\varepsilon)$. An application is given to the Bernoulli-Graeffe method for approximation of the roots of algebraic equations. *N. G. de Bruijn* (Amsterdam)

670:

Pitman, Jane. The inhomogeneous minima of a sequence of symmetric Markov forms. Acta Arith. **5** (1958), 81–116 (1959).

Using the 'algorithm of the divided cell' as introduced by Delone [Izv. Akad. Nauk SSSR. Ser. Mat. **11** (1947), 505–538; MR **9**, 334] and elaborated by Barnes and Swinnerton-Dyer [Acta Math. **92** (1954), 199–234; MR **16**, 802] and Barnes [Acta Math. **92** (1954), 235–264; MR **16**, 802], the author determines the inhomogeneous minimum of a certain sub-class of the Markoff binary quadratic forms, and in some cases also the second minimum. The results show that the lower bound of the expression

(inhomogeneous minimum)/(determinant)$^{\frac{1}{2}}$

over all indefinite quadratic forms can be at most equal to 1/12, which is a new low. Together with the recent result of Ennola [Ann. Univ. Turku Ser. A I **28** (1958); MR **20** #3825] this gives a much narrower range in which this lower bound must lie. The paper contains an almost complete account of the algorithm of the divided cell. {Miss Pitman has since shown that the lower bound in question is definitely smaller than 1/12, but has not yet published her work.}

J. W. S. Cassels (Cambridge, England)

FIELDS

671:

Morikawa, Hisasi. Correction to my paper "On the existence of unramified separable infinite solvable extensions of function fields over finite fields". Nagoya Math. J. **14** (1959), 53–57.

Nine necessary technical lemmas are added to the proof of the title result of the author's previous paper [same J. **13** (1958), 95–100; MR **20** #2329].

M. Rosenlicht (Evanston, Ill.)

672:

Morikawa, Hisasi. On unramified separable abelian p-extensions of function fields. I. Nagoya Math. J. **14** (1959), 223–234.

Let K be a function field of one variable over the algebraically closed field k of characteristic $p > 0$, L/K an unramified abelian extension of degree a power of p, $G(L/K)$ the galois group. $G(L/K)$ has a representation as automorphisms of the jacobian variety J_L, hence (using the points of J_L of order a power of p) a representation by p-adic integral matrices. This representation is proved to be equivalent (as a representation over the p-adic numbers, indeed over the p-adic integers if L/K is cyclic) to $(\gamma - 1)$ times the regular representation plus the identity representation, p^{ν} being the number of x on J_K satisfying $px = 0$. *M. Rosenlicht* (Evanston, Ill.)

673:

★de Boer, J. H. The multiplicity theory for specialized fields. Getal en Figuur, 9. Van Gorcum & Co. N. V.-G. A. Hak & Dr. H. J. Prakke, Assen, 1958. i+39 pp.

This paper contains some extensions of Krull's theorems on the Hilbert and ramification theory of valuations. A modification of the results of I. S. Cohen and O. Zariski [Illinois Math. **1** (1957), 1–8; MR **18**, 788] pertaining to the relation $\sum e_i f_i \leqq n$ is at the center of the author's discussion

O. F. G. Schilling (Chicago, Ill.)

LINEAR ALGEBRA

See also 655, 760, 896, 965.

674:

Pearl, Martin. A further extension of Cayley's parameterization. Canad. J. Math. **11** (1959), 48–50.

The author proposes to generalize theorem 1 [or 4] of his preceding paper [Canad. J. Math. **9** (1957), 553–562; MR **19**, 936], taking instead of the real [or complex] field an arbitrary field F of characteristic $\neq 2$. Assuming in F an involutory automorphism $\lambda : a \to \bar{a}$, the new theorem 1′ coincides verbally with theorem 1, where A is now a matrix in F, hermitean with respect to λ, and the condition $|P| = 1$ is replaced by the condition $|P| = a/\bar{a}$ $(a \neq 0)$. It is not stated under what conditions λ may be replaced by the identity in F (which is done in order to obtain the former theorem 1, but certainly cannot be done if F is the complex number field). The author's proof of theorem 1′ is merely sketched and the reviewer is unable to follow the outline. *H. Schwerdtfeger* (Montreal, P.Q.)

675:

Zassenhaus, Hans. On a normal form of the orthogonal transformation. I, II, III. Canad. Math. Bull. **1** (1958), 31–39; 101–111; 183–191.

The papers deal with an answer to a question raised by Wigner concerning the characteristic roots of the linear automorphs of the quadratic form $t^2+x^2-y^2-z^2$. For this purpose "generalized orthogonal" transformations over a field F are studied and normal forms obtained for them. The field F is assumed perfect and of characteristic different from 2. The normal form exhibits the characteristic polynomial. Special applications are made to the real field and to Galois fields. {The author informed the reviewer that Williamson [Amer. J. Math. **61** (1939), 335–356] had solved the same problem (even for normal matrices) by a method which leads to different normal forms. He uses a matrix approach while Zassenhaus uses decomposition of linear spaces.} *O. Taussky-Todd* (Pasadena, Calif.)

676a:

Schwerdtfeger, H. On the discriminant $x'Ax\cdot y'Ay-(x'Ay)^2$. Canad. Math. Bull. **1** (1958), 175–179.

676b:

Scherk, Peter. On a note by H. Schwerdtfeger. Canad. Math. Bull. **1** (1958), 181–182.

Joining a result of the reviewer [Uspehi Mat. Nauk **11** (1956), no. 3 (69), 3–68; MR **18**, 807; p. 42; cf. also J. Aczél and O. Varga, Publ. Math. Debrecen **4** (1955), 3–15; MR **17**, 777; p. 5] who proved the implication (i)⇒(ii) (see below), the present authors announce that the following two properties of a regular real symmetric matrix A are equivalent: (i) A is of the congruence type $[+,-,\cdots,-]$; (ii) $y'Sy=x'Ax\cdot y'Ay-(x'Ay)^2\leq 0$ for all real column vectors y, equality holding if and only if y is a multiple of x (x is a fixed real column vector such that $x'Ax>0$). The second author's proof is shorter while the first author discusses also the eigenvalues of $S=x'Ax\cdot A-Axx'A$.

J. Aczél (Debrecen)

677:

Wild, Jonathan. On the index of a quadratic form. Canad. Math. Bull. **1** (1958), 180.

Given a vector space $V=\{x, y, \cdots\}$ over an arbitrary field and a symmetric bilinear form $f(x, y)$ in it, a subspace W is called totally isotropic if $f(x, y)=0$ for every pair $x\in W, y\in W$. The author gives a simple direct proof of the well-known fact, that for two subspaces V_n, V_m of V having dimensions $n<m$ respectively, there exists a totally isotropic subspace W_m of dimension m containing V_n. Two cases as to whether or not V_n and V_m have common points are considered. *J. Aczél* (Debrecen)

678:

Nakamori, Kanzi. A theory of p-dimensional determinants. Yokohama Math. J. **6** (1958), 79–88.

A determinant of order n as usually understood is, for the purposes of this note, a two-dimensional determinant. From the expression for such a determinant in terms of the Kronecker delta, the author obtains a generalization which he calls a p-dimensional determinant. He generalizes the classical properties of determinants, giving a multiplication law, introducing minors, co-factors and the Laplace expansion. He ends with a generalization of a theorem due to Sylvester. *E. T. Davies* (Southampton)

679:

Smith, Richard B. Two theorems on inverses of finite segments of the generalized Hilbert matrix. Math. Tables Aids Comput. **13** (1959), 41–43.

Let $h_{ij}=1/(p+i+j-1)$ $(i, j=1, 2, \cdots, n;$ p not an integer in the range -1 to $-(2n-1))$ be the elements of the generalized Hilbert matrix and denote by S^{ij} the elements of its inverse. The author shows that $\sum_{i=1}^{n}\sum_{j=1}^{n}S^{ij}=n(p+n)$ and gives also a formula for the row sums $\sum_{j=1}^{n}S^{ij}$. *E. Lukacs* (Washington, D.C.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 682, 805.

680:

Jones, A.; and Schiffer, J. J. Concerning the structure of certain rings. Bol. Fac. Ingen. Agrimens. Montevideo **6** (1957/58), 327–335. Also published as Fac. Ingen Montevideo. Publ. Inst. Mat. Estadist. **3** (1958), 81–89. (Spanish. English summary)

All rings such that every additive subgroup is a two-sided ideal are determined.

M. Henriksen (Lafayette, Ind.)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

681:

Albert, A. A. A solvable exceptional Jordan algebra. J. Math. Mech. **8** (1959), 331–337.

It is proved that the product $\mathfrak{G}=\mathfrak{H}\times\mathfrak{N}$ of a simple exceptional Jordan algebra $\mathfrak{H}$ over a field $\mathfrak{F}$ of characteristic not 2 and the associative polynomial algebra $\mathfrak{N}=\mathfrak{F}(g)$ generated by a nilpotent element g of index at least 33, is an exceptional solvable Jordan algebra. The question of the existence of such algebras had been raised by N. Jacobson. *J. L. Tits* (Brussels)

682:

Zemmer, Joseph L. Some $\mathfrak{G}$ division algebras. Canad. J. Math. **11** (1959), 51–58.

An algebra over a field F, which possesses an associative bilinear form $h(x, y)$ and satisfies $(x\cdot y)\cdot z-x\cdot(y\cdot z)=h(y, z)x-h(x, y)z$ is called a $\mathfrak{G}$ algebra. Let K^* be an associative algebra over F with identity u and a basis $u, e_1, e_2, \cdots$. Then, for x, y in K (the space spanned by $e_1, e_2, \cdots$), $xy=\alpha u+a$, $a\in K$. Let $h(x, y)=\alpha$ and $x\cdot y=a$. K with the operation $\cdot$ is then a $\mathfrak{G}$ algebra. K^* is called its associated associative algebra. Any $\mathfrak{G}$ algebra can be thus obtained from a unique associative algebra with an identity. If $K(\circ)$ and $L(*)$ are $\mathfrak{G}$ algebras then their associated associative algebras K^* and L^* are equivalent if and only if there exists a one–one linear mapping T of K onto L and a linear functional f on K such that

$$(x\circ y)T=f(x)(yT)+f(y)(xT)+xT*yT$$

for all x, y in K. K and L are said to be pseudo-equivalent.

If K is a $\mathfrak{G}$ division algebra ($2<\dim K<\infty$) then either its associated associative algebra K^* is a division algebra or $K^*=Fu^*\oplus A^*$, where A^* is a division algebra pseudo-equivalent to K.

Let $\lambda^3-\tau(x)\lambda^2+\alpha(x)\lambda-\nu(x)=0$ be the equation satisfied by $x\in A$, a cyclic division algebra of degree 3 over a field F of characteristic $\neq 2$. Let $f(x)=-\frac{1}{3}[\tau(x)]$ and $A(\circ)$ be the pseudo-equivalent algebra with $\circ$ defined by $x\circ y=f(x)y+f(y)x+xy$. Then $A(\circ)$ is a division algebra. The isotope $A(*)$, defined by $x*y=xU^{-1}\circ yU^{-1}$ where $xU=x\circ 1$ (1 the identity in A), is central and contains a subalgebra isomorphic to a certain three-dimensional algebra of Dickson. *H. Minc* (Vancouver, B.C.)

GROUPS AND GENERALIZATIONS

See also 837, 838, 844, 950.

683:

Piccard, Sophie. Les groupes que peuvent engendrer trois éléments a, b, c générateurs d'un groupe multiplicatif, qui satisfont le système non exhaustif de relations fondamentales $a^2=1$, $b^2=1$, $c^2=1$, $(ab)^3=1$, $(ac)^3=1$, $(bc)^3=1$. Publ. Sém. Géom. Univ. Neuchâtel **1** (1958), no. 1, 42 pp.

Groups of the sort described in the title are examined, first putting aside the trivial case that the group is generated by two of a, b, c, and is hence a homomorphic image of S_3. Each element can be written in one of the forms $x(caba)^k(cab)^ky$, $y=1$, c, ca, or $x(caba)^k(cba)^kz$, $z=1$, c, cb, where $x\in\{a,b\}$. The elements cab and cba will have the same order $s\leqq\infty$, and $caba$ will have order r satisfying $2r=s$ or $2r=3s$. If the group is finite, it has order $3rs$, and can be generated by two elements if and only if r is not divisible by 3. Groups of permutations of the integers of each of these finite types, and one of infinite order, are exhibited. *R. C. Lyndon* (Ann Arbor, Mich.)

684:

Piccard, Sophie. Structure des groupes libres. I, II, III. C. R. Acad. Sci. Paris **248** (1959), 343–346, 508–511, 623–626.

The author gives various special results concerning free groups which are for the most part well known and can easily be proved directly. E.g., a free group F of infinite rank m has more than m normal subgroups. The results are proved by considering homomorphisms of F (onto abelian groups), which are described at great length.

P. M. Cohn (Manchester)

685:

Sholander, Marlow. Postulates for commutative groups. Amer. Math. Monthly **66** (1959), 93–95.

The more striking of two results asserts that a system closed under a binary operation $x-y$ satisfying the identity $y=x-[(x-z)-(y-z)]$ is an abelian group.

R. C. Lyndon (Ann Arbor, Mich.)

686:

Plotkin, B. I. Generalized solvable and generalized nilpotent groups. Uspehi Mat. Nauk **13** (1958), no. 4 (82), 89–172. (Russian)

The paper is a continuation of the survey "Soluble and nilpotent groups" by Kuroš and Černikov [same Uspehi **2** (1947), no. 3 (19), 18–59; Amer. Math. Soc. Transl. no. 80 (1953); MR **10**, 677; **14**, 618] and sketches without proofs the contributions to the subject, as far as they are not yet covered by the second edition of the book *Teoriya grupp* by Kuroš [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953; English translation in 2 vols., Chelsea, New York, 1955/56; MR **15**, 501; **17**, 124; **18**, 188]. The bibliography, which goes to the end of 1957, lists just over 200 items, a sign of the vigorous development of this branch of the theory of groups; of these 120 items are by Russian authors. The five chapters of the paper deal with general concepts, automorphism groups, generalised nilpotent groups, generalised soluble groups, and the lattice of subgroups of generalised soluble and nilpotent groups. An English translation of the valuable paper is in preparation.

K. A. Hirsch (London)

687:

Schreiber, Shmuel. On two conjectures in the theory of finite groups. Riveon Lematematika **12** (1958), 73–75. (Hebrew. English summary)

Let r and s be two real numbers, not both equal to 1, such that $r+s$ and rs are coprime integers; let n be an integer greater than 2. Carmichael has made the following two conjectures: (1) Let $D_n=(r^n-s^n)/(r-s)$ and suppose that D_n has a prime factor q that is not a factor of any D_i with $1\leqq i\leqq n-1$. Then a group of order D_n has a normal subgroup of order q^α, $\alpha>0$. (2) Let $F_n=s^{\nu(n)}\cdot\Phi_n(r/s)$, where Φ_n denotes the nth cyclotonic polynomial. Then a group of order F_n has a normal subgroup of order q^α $(\alpha>0)$, where q is a prime factor of F_n, but not of n. The author shows that both conjectures are false, even for $n=3$. In fact he gives an easy method of constructing an infinite number of counter examples. For instance, $r+s=7$, $rs=-11$ gives $D_3=60$ and $r+s=53$, $rs=829$ gives $D_3=2520$. *K. A. Hirsch* (London)

688:

Gorenstein, Daniel. A class of Frobenius groups. Canad. J. Math. **11** (1959), 39–47.

A finite group G is called an independent ABA-group if it has two subgroups A, B such that every element not in A is uniquely of the form aba' with a, a' in A and b in B. It is proved that an independent ABA-group is a Frobenius group with A as subgroup of stability; and that if the orders of A and the regular subgroup M are h and m, respectively, then if we put $k=1+(m-1)/h$, k divides the order hm of G. Moreover, if G is a Frobenius group satisfying this arithmetic condition, then its regular subgroup is either elementary abelian, or a direct product of two cyclic groups of order 4, or a direct product of two elementary abelian groups whose orders satisfy a certain relation. This enables the author to identify the independent ABA-groups: they are the one-dimensional affine groups over near-fields, and other related groups.

Graham Higman (Oxford)

689:

Smith, G. F.; and Rivlin, R. S. The anisotropic tensors. Quart. Appl. Math. **15** (1957), 308–314.

The authors consider tensors in Euclidean 3-space with Cartesian rectangular coordinates. A tensor is called isotropic if its coordinates are invariant under an arbitrary orthogonal transformation; in the contrary case the tensor is anisotropic.

It is shown that every anisotropic tensor with coordinates invariant with respect to a subgroup T of the group of orthogonal transformations is a linear combination with scalar coefficients of products of a finite number of

tensors. The tensors in question are found for the following cases: T is the group associated with crystals of the "domatic" class of the monoclinic system; T is a group of this kind corresponding to the pyramidal-rhombic class of the rhombic system; the subgroup T consists of rotations about the straight line h and of reflections in the plane perpendicular to h. *G. B. Gurevič* (RŽMat **1959** #815)

690:

Karzel, Helmut. Kennzeichnung der Gruppe der gebrochen-linearen Transformationen über einem Körper der Charakteristik 2. Abh. Math. Sem. Univ. Hamburg **22** (1958), 1–8.

F. Bachmann [Math. Ann **126** (1953), 79–92; MR **15**, 197] has given a characterisation of the 1-dimension projective group $\{X'=(AX+B)(CX+D)^{-1},\ AD-BC\neq 0\}$ over a field of characteristic $\neq 2$, in terms of involutory elements. The same is done here for the characteristic 2 case. The axiom system, based on E. Sperner's axiom system for the absolute geometry [Arch. Math. **5** (1954), 458–468; MR **16**, 278. Cf. also H. Karzel, ibid. **6** (1955), 66–76, 284–295; MR **16**, 395; **17**, 776] is as follows. Let $\mathfrak{G}$ be a group in which every element is expressible as a product of two involutory elements, let $\mathfrak{E}$ be the set of all involutory elements in $\mathfrak{G}$ and $\mathfrak{R}=\mathfrak{G}-\mathfrak{E}-\{1\}$. $\alpha, \beta \in \mathfrak{G}$ are "bound" (verbindbar) if $\exists u \in \mathfrak{E}$ such that $(u\alpha)^2=(u\beta)^2=1$. Ax. I: If $a\neq b$, and $a, b, c, d, abc, abd \in \mathfrak{E}$ then $(acd)^2=1$. Ax. II: There exist $\alpha \in \mathfrak{R}$ and $a \in \mathfrak{E}$ that are not bound. Ax. III: If $a_1, a_2, a_3 \in \mathfrak{E}$ are pairwise not bound, every $\alpha \in \mathfrak{R}$ is bound with at least one a_i; if $\alpha \in \mathfrak{R}$ is bound neither with a_1 nor with a_2, then it is with a_1a_2. Ax. IV: If a is bound with b and with c $(a, b, c \in \mathfrak{E})$, then $(abc)^2=1$.

J. L. Tits (Brussels)

691:

Rzewuski, Jan. The 13-parametric group of transformations of the spinor space and its representations. Acta Phys. Polon. **17** (1958), 417–428. (Russian summary)

The author considers a 13-dimensional Lie group of transformations of spinor space, writes down the commutation relations for the corresponding Lie algebra and relates them to physical concepts. The author claims to have constructed all irreducible representations in this article; the reviewer can find no evidence for this.

F. I. Mautner (Paris)

692:

Comét, Stig. Über die Anwendung von Binärmodellen in der Theorie der Charaktere der symmetrischen Gruppen. Numer. Math. **1** (1959), 90–109.

The author develops a scheme for describing sets of objects by numbers which in its simplest form is as follows: The objects are numbered $0, 1, 2, \cdots$. A set of objects with numbers $m, n, r, \cdots$ is assigned the number $2^m+2^n+2^r+\cdots$. By means of this correspondence certain formulas and theorems concerning the characters of the symmetric group are derived, in particular a formula of I. Schur. *M. Newman* (Washington, D.C.)

693:

Nagao, Hirosi. A remark on the orthogonality relations in the representation theory of finite groups. Canad. J. Math. **11** (1959), 59–60.

The orthogonality relations between the coefficients of the absolutely irreducible representations of a finite group are derived from the character relations $\sum_\alpha f_\alpha \chi^\alpha(t)=\delta_{1,t}g$.

H. K. Farahat (Sheffield)

694:

Winogradzki, J. La représentation spinorielle du groupe de Lorentz général. Cahiers de Phys. **12** (1958), 261–272.

The spin representation S of the general Lorentz group L is defined by means of two conditions, of which the first ensures the relativistic invariance of the Dirac wave equation, while the second enables one to evaluate an arbitrary coefficient of S which appears when S is derived in accordance with the first condition. In this manner explicit forms of S are deduced, from which the fundamental properties of S are directly obtained: it is irreducible and unimodular. Special attention is paid to the representations which are characterized by the unitary and hermitian Dirac matrices. The subgroups (of the general Lorentz group) containing the group of spatial rotations for which the representation is reducible are determined, and the corresponding representation for each of these subgroups is indicated. Amongst these representations are some which had been given previously [Winogradzki, C. R. Acad. Sci. Paris **245** (1957), 2206–2208; MR **19**, 994; de Broglie, *Théorie générale des particules à spin*, Gauthier-Villars, Paris, 1954].

H. Rund (Durban)

695:

Kurepa, Svetozar. Semigroups of linear transformations in n-dimensional vector space. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II. **13** (1958), 3–32. (Serbo-Croatian summary)

Let G_+ denote the semi-group of all rational numbers of the form $l/2^k$ $(l, k=0, 1, 2, \cdots)$ under addition. Let $\Phi(r)$ be a function defined on G_+ with values in the space of linear transformations (l.t.) on an n-dimensional vector space R. If $\Phi(r+r')=\Phi(r)\Phi(r')$, for all $r, r' \in G_+$, then $\Phi(r)$ is called a representation of the semi-group G_+ as a semi-group of l.t. The author makes a thorough study of such representations. The main theorem of the paper asserts that "If $\Phi(r)f=0$ implies $f=0$, for every $r \in G_+$, then there exists a basis $f_1, \cdots, f_n$ in R such that $\Phi(r)$ has the form

$$\overline{\Phi(r)} = \exp(r(\overline{A}+\overline{B}))\cdot\overline{U(r)} = \overline{U(r)}\cdot\exp(r(\overline{A}+\overline{B})) \qquad (r \in G_+),$$

where $\overline{A}$ denotes the matrix form of the l.t. A with respect to the basis, $\overline{A}^n=0$, $\overline{B}$ is a diagonal hermitian matrix, and $\overline{U(r)}$ is a semi-group of diagonal unitary matrices." A similar result is also obtained for the m-parameter case. If $\Phi(2^{-k})\to E$ (the identity l.t.) as $k\to\infty$, then there exists a unique l.t. Φ such that $\Phi(r)=\exp(r\Phi)$. These results are extended, under certain additional conditions, to representations of the semi-group of all non-negative real numbers.

C. E. Rickart (New Haven, Conn.)

696:

Sade, A. Groupoïdes automorphes par le groupe géométrique et quasigroupes "endo". Canad. J. Math. **10** (1958), 294–320.

The paper consists of a multitude of examples and a variety of elementary theorems concerning structures of the following type: R is a specifically given associative

ring, Q is a groupoid consisting of the elements of R under some operation not specified in advance, M is a non-empty subset of R, and the product xm of the element x of Q and the element m of M is to be computed using the multiplicative operation of R. The only condition imposed upon Q is that the mappings $x \to xm$ be endomorphisms of Q.

Let us consider the case that R is the ring of integers modulo n. Then, according as M is a multiplicative subsemigroup of R or consists of all integers prime to n, Q is said to be "endo" or "automorphic under the geometric group". (Somewhat more general definitions of these concepts are given but are not much used.)

The paper is similar in spirit to an earlier one by the author [same J. **9** (1957), 321–335; MR **19**, 389].

R. H. Bruck (Madison, Wis.)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 691, 694.

697:

Beck, Anatole; Corson, H. H.; and Simon, A. B. The interior points of the product of two subsets of a locally compact group. Proc. Amer. Math. Soc. **9** (1958), 648–652.

Let G be a locally compact group with completed Haar measure m and outer measure m^*. Let A and B be subsets of G such that $m(A) > 0$ and $m^*(B) > 0$. The basic result of this note is: (1) The set AB of products has nonvacuous interior. This has a number of consequences for subsemigroups of G. (2) Any subsemigroup of G which contains a set of positive measure contains an open subsemigroup. (3) If G is compact, any subsemigroup which contains a set of positive measure is an open subgroup of G. (4) If G is compact and abelian, every maximal semigroup is either a group or is not measurable. (5) In any locally compact G, if S is a measurable semigroup, if T is a measurable group, and if $m(S \Delta T) < m(T)/2$, then $S = T$. (This last says that a semigroup which is more than half of a group is in fact a group.) The basic theorem (1) is the measure theoretic analogue of a similar result on sets of second category, due to McShane [Ann. of Math. (2) **51** (1950), 380–386; MR **12**, 518]. *F. B. Wright* (New Orleans, La.)

698:

Gluškov, V. M. Structure of locally bicompact groups and Hilbert's fifth problem. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 2 (74), 3–41. (Russian)

This is essentially an expository paper on the well-known results of Montgomery, Zippin, Gleason and Yamabe [cf. for instance the bibliography in D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, New York, 1955; MR **17**, 383], which, however, includes several interesting new remarks. In particular, the final structure theorem is given the following form, somewhat more general than Yamabe's: (A) For any locally compact group G and any neighborhood of the identity U, there exists an open neighborhood of the identity $V \subset U$ which is the direct product of a local Lie group and a compact group; if G is not totally disconnected, U can be chosen such that for any such decomposition, the local Lie group has positive dimension. For connected L-groups, this local decomposition property was known to Iwasawa [Ann. of Math. (2) **50** (1949), 507–558; MR **10**, 679]. In order to prove (A), the author has to get rid of the connectedness assumption, which he achieves by using a method due to Malcev [Mat. Sb. (N.S.) **19 (61)** (1946), 165–174; MR **8**, 439].

J. L. Tits (Brussels)

699:

Gluškov, V. M. Lie algebras of locally bicompact groups. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 2 (74), 137–142. (Russian)

Let G be a topological group and let $A(G)$ denote the set of all one-parameter subgroups of G (i.e. continuous homomorphisms $R \to G$). The following operations are introduced in $A(G)$:

$$(r \cdot x)(s) = x(rs)$$

$$(x+y)(s) = \lim_{n\to\infty} \left(x\left(\frac{s}{n}\right)y\left(\frac{s}{n}\right)\right)^n$$

$$[x, y](s) = \lim_{n\to\infty} \left(x\left(-\frac{s}{n}\right)y\left(-\frac{s}{n}\right)x\left(\frac{s}{n}\right)y\left(\frac{s}{n}\right)\right)^{n^2}.$$

If these operations exist for all x, y, and if they define a Lie algebra structure on $A(G)$, G is called quasi-Lie. Such is every locally compact group, as follows easily from Yamabe's result [see the preceding review]. $A(G)$ is further given a natural topology (which does not necessarily make $x+y$ and $[x, y]$ continuous). The following theorems are proved for an arbitrary locally compact group G: The natural homomorphism of G on any factor group G/N induces a topological isomorphism of $A(G)/A(N)$ on $A(G/N)$ (this is not true for all quasi-Lie groups; in fact, according to the author, there exist a quasi-Lie group G with a quasi-Lie factor group G/N such that $A(G) = \{0\} \neq A(G/N)$). $A(G)$ is the topological product of finite dimensional Lie algebras, all but a finite number of which are compact or abelian; conversely, every such product is the Lie algebra of some locally compact group. If G is connected, the natural homomorphism of the group of automorphisms of G in the group of automorphisms of $A(G)$ is injective.

J. L. Tits (Brussels)

700:

Dobrescu, Andrei. Sur l'équation caractéristique d'un groupe G_r. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **5** (1956), no. 9, 25–29. (Romanian. Russian and French summaries)

A proof is supplied for the fact, merely stated in E. Cartan's thesis [Nony, Paris, 1894; p. 106; *Œuvres complètes*, partie I, vol. I, Gauthier-Villars, Paris, 1952; MR **14**, 343; p. 236], that the sum of the p-th powers of the roots of the characteristic equation $|c_{ik}{}^h e^s - \delta_k{}^h \rho| = 0$ of an r-dimensional Lie algebra is equal to

$$c^{\lambda_1}_{h_1\lambda_2} c^{\lambda_2}_{h_2\lambda_3} \cdots c^{\lambda_p}_{h_p\lambda_1} e^{h_1} e^{h_2} \cdots e^{h_p} \quad (p = 1, \cdots, r).$$

J. L. Tits (Brussels)

701:

Dobrescu, Andrei. Un critère concernant les groupes de rang zéro. Com. Acad. R. P. Romîne **7** (1957), 923–927. (Romanian. Russian and French summaries)

The following is proved: an r-dimensional Lie algebra with structure constants $c_{kl}{}^h$ has rank 0 if and only if $c_{h_1h_2\cdots h_s} = c^{\lambda_1}_{h_1\lambda_2} c^{\lambda_2}_{h_2\lambda_3} \cdots c^{\lambda_s}_{h_s\lambda_1} = 0$ for all $s = 1, \cdots, r-1$.

J. L. Tits (Brussels)

702:

Dieudonné, Jean. Sur les groupes formels abéliens unipotents. Rend. Circ. Mat. Palermo (2) **5** (1956), 170–180.

Let G be a formal Lie group over a field K of characteristic $p>0$, and let $z=\varphi(x, y)$ be the law of composition in G. For each integer $r>1$, define $\varphi^{(r)}(x)$ inductively by the formula $\varphi^{(r)}(x)=\varphi(x, \varphi^{(r-1)}(x))$, where $\varphi^{(1)}(x)=x$. The group G is said to be unipotent of exponent r if the n formal series $\varphi_i^{(r)}(x)$ $(1\leq i\leq n=$ dimension of $G)$ are identically zero. The exponent of a unipotent group is always a multiple of p. The author determines the abelian unipotent groups of exponent a power of p, using a previous result [Amer. J. Math. **77** (1955), 429–452; MR **17**, 174] to the effect that an abelian group of dimension n is isomorphic to an abelian group associated in a certain way with an n by n matrix U with coefficients belonging to the ring $\mathfrak{o}^+(K)$ defined in the above paper. The group G is not determined uniquely by the matrix U, but only modulo an equivalence relation. The author's main theorem asserts that over a perfect field K, the abelian formal Lie group G is unipotent of exponent p^r if and only if G is isomorphic to a group associated with a matrix U such that

$$U^{\sigma^{r-1}}U^{\sigma^{r-2}}\cdots U^{\sigma}U=0,$$

where σ is an automorphism of the ring $\mathfrak{o}^+(K)$ which reduces to the identity if K is the prime field of p elements. The proof of the theorem as well as several other results in the paper are based on previous works of the author on the subject. *C. W. Curtis* (Madison, Wis.)

FUNCTIONS OF REAL VARIABLES

See also 961.

703:

Zandere, M. K. Partial continuity of functions of several variables. Latvijas Valsts Univ. Zinātn. Raksti **20** (1958), no. 3, 47–55. (Russian. Latvian summary)

The author considers functions $f(x_1, \cdots, x_n)$, of n real variables, that are continuous along each curve belonging to a given system. He finds certain rather complicated sufficient conditions under which such a function is totally continuous (that is, continuous in all its arguments jointly) at a set of points everywhere dense in a given domain, and deduces the following theorem for a function that is "linearly continuous", that is, continuous in each of its arguments separately: on every set homoeomorphic to an $(n-1)$-dimensional cube the set of points of (total) continuity of a linearly continuous function of n arguments is everywhere dense. This generalizes a result given by H. Hahn [Math. Z. **4** (1919), 306–313].

H. P. Mulholland (Exeter)

704:

Pettineo, Benedetto. Quelques observations sur les fonctions dérivables presque partout. C. R. Acad. Sci. Paris **248** (1959), 518–520.

This note is a continuation of a previous note of the author [same C. R. **243** (1956), 553–554; MR **18**, 876]; the same notations are used. The author states the following conjecture: Every continuous function $F(x)$ which is determined on $[a, b]$ (up to a constant) by its derivative $f(x)$, existing almost everywhere, has the property A: If F is of bounded variation on the closed set C, it is also absolutely continuous on C such that for the variation $\nu[C]$ of F on C one has $\nu[C]=\int_C f(x)dx$. The author states that every integral defined up to now (in particular the special Denjoy integral) has this property A. Moreover he states and (for the variation defined by A. Denjoy) proves the converse: Every continuous function F having the property A is determined on $[a, b]$ (up to a constant) by its derivative f, existing almost everywhere.

A. Rosenthal (Lafayette, Ind.)

705:

Hart, William L.; and Motzkin, Theodore S. Proof of the fundamental theorem on implicit functions by use of composite gradient corrections. Pacific J. Math. **8** (1958), 429–436.

The authors give a constructive type of proof of the classical implicit function theorem. The proof is based on an iterative gradient method described by the authors in a previous paper [same J. **6** (1956), 691–707; MR **19**, 176].

The theorem in question concerns the existence of n functions $x_i(t)$ $(i=1, \cdots, n)$, which satisfy a system of n equations $f_j(x; t)=0$ $(j=1, \cdots, n)$ in the n variables $(x_1, \cdots, x_n)=x$ and the p variables $(t_1, \cdots, t_p)=t$. The usual hypotheses are made, namely, that the f_j are continuous and all partial derivatives $\partial f_j/\partial x_i$ exist and are continuous in some neighborhood of the point $x=\alpha$, $t=\beta$, $f_j(\alpha; \beta)=0$ for $j=1, \cdots, n$, and the matrix $A=(a_{ij})$, $a_{ij}=\partial f_j(\alpha; \beta)/\partial x_i$, is non-singular.

Referring to their previous paper, the authors introduce the eigenvalues, λ_i, of the matrix AA'. Letting $\sigma_\rho=\mathrm{Max}_i\{1-\rho\lambda_i\}$, they give sufficient conditions on ρ to insure that $\sigma_\rho<1$. For such a ρ, they define a sequence, $x^{(m)}(t)$, by the recursion, $x^{(0)}(t)=\alpha$, $x^{(m)}(t)=x^{(m-1)}(t)+\Delta x^{(m-1)}(t)$, where the ith component of $\Delta x^{(m-1)}$ is given by

$$\Delta x_i^{(m-1)} = -\rho \sum_{j=1}^{n} \left\{ f_j(x^{m-1}; t)a_{ij} \Big/ \sum_{i=1}^{n} a_{ij}^2 \right\}.$$

Under the stated hypotheses, it is shown that there exist $\varepsilon>0$ and $\delta>0$, $\delta\leq\varepsilon$, such that if $\|t-\beta\|\leq\delta$, all of the $x^{(m)}(t)$ exist, are continuous and satisfy $\|x^{(m)}(t)-\alpha\|\leq\varepsilon$. Also, for $\|t-\beta\|\leq\delta$, the $x^{(m)}(t)$ converge uniformly to $x(t)=(x_1(t), \cdots, x_n(t))$, the desired solution of $f_j(x; t)=0$. Further, if a point $(x; t)$ with $\|x-\alpha\|\leq\varepsilon$ and $\|t-\beta\|\leq\delta$ satisfies $f_j(x; t)=0$, then $x=x(t)$. *E. K. Blum* (Los Angeles, Calif.)

706:

Călugăreanu, G. Sur la représentation analytique des régions du plan. Rev. Math. Pures Appl. **2** (1957), 281–288.

The reviewer found this paper hard to read. It is concerned with C' mappings f of a closed region $\bar{D}$ into the plane for which the Jacobian is not identically zero. By an integration process an analytic representation for the image of $\bar{D}$ is obtained. In terms of this representation various sets of conditions are obtained that the mapping f have maximum valence $\leq p$.

C. J. Titus (Ann Arbor, Mich.)

707:

Obrechkoff, Nikola. Sur une généralisation du théorème de Poulain et Hermite pour les zéros des polynomes réels. C. R. Acad. Bulgare Sci. **11** (1958), 5–8. (Russian summary)

Let

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

have only real zeros and let the real polynomial

$$g(z) = b_0x^m + b_1x^{m-1} + \cdots + b_m, \quad b_m \neq 0,$$

have the real zeros $x_1, x_2, \cdots, x_k$ and the non-real zeros $r_se^{\pm i\phi_s}$, $s = 1, 2, \cdots, p$, $k + 2p = m$. The author proves that if $|\sin\phi_s| \leq 1/\sqrt{n}$, $s = 1, 2, \cdots, p$, then all zeros of $F(x) = g(D)f(x)$, $D = d/dx$, are real and, if $|\sin\phi_s| < 1/\sqrt{n}$, each multiple zero of $F(x)$ is a multiple zero of $f(x)$. This result reduces to the Poulain-Hermite theorem when each $\phi_s = 0$ and thus $F_1(x) = h(D)f(x)$ has only real zeros if $h(x) = (x - x_1)(x - x_2)\cdots(x - x_k)$. The result is proved by showing that

$$F_{j+1}(x) \equiv F_j(x) - 2r_j\cos\phi_jF_j'(x) + r_j^2F_j''(x)$$

for $j = 1, 2, \cdots, p$ has likewise only real zeros.

M. Marden (Milwaukee, Wis.)

MEASURE AND INTEGRATION

See also 663, 664, 808.

708:

Kurepa, Svetozar. On the (C)-property of functions. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II. **13** (1958), 33–38. (Serbo-Croatian summary)

Let $\Phi(t)$ be a function defined on a finite closed interval δ and with values in the space of all bounded linear operators on a Hilbert space R. $\Phi(t)$ is weakly continuous [weakly measurable] if the function $(\Phi(t)f, g)$ is continuous [measurable], for all $f, g \in R$. The function is said to possess the (C)-property on δ if, for arbitrary $\varepsilon > 0$, there exists a perfect set $P \subseteq \delta$ such that $m(P) > m(\delta) - \varepsilon$ and the function $\Phi(t)$ is weakly continuous on P. This property was studied by Luzin [C. R. Acad. Sci. Paris **154** (1912), 1688–1690] for R one-dimensional. The author extends some of the Luzin results to the present situation. For example, if R is separable, then $\Phi(t)$ has the (C)-property if and only if it is weakly measurable. On the other hand, if R is not separable, then there exists an example of a function which is weakly measurable but does not have the (C)-property.

C. E. Rickart (New Haven, Conn.)

709:

Alfsen, Erik M. On a general theory of integration based on order. Math. Scand. **6** (1958), 67–79.

The author attempts a generalization of the extension theory of integration, depending only upon the order properties of function spaces. Let L be a lattice. A real-valued function I over L is called an 'integral', if (I) $x \geq y$ implies $I(x) \geq I(y)$; (II) $I(x) + I(y) = I(x \cup y) + I(x \cap y)$; (III) $\sup_n I(y_n) \geq \inf_n I(z_n)$, if $(y_n \uparrow) \geq (z_n \downarrow)$, that is, if $y \geq y_n$, $z \leq z_n$ for all n implies $y \geq z$. An integral I over L is said to be 'full', if (IV) $x_n \uparrow$, $\sup_n I(x_n) < \infty$ implies $x_n \uparrow x \in L$. The author proves an extension theorem: If L is a sublattice of a σ-complete lattice H satisfying the countable distributivity

$$(\bigcup_n x_n) \cap (\bigcup_m y_m) = \bigcup_{n,m}(x_n \cap y_m),$$

then for any integral I over L we can extend I over a sublattice $\bar{L}$ of H such that $L \subset \bar{L}$ and I is full over $\bar{L}$.

{The author states that this theorem is a generalization of the extension theory of integration, but in truth it is not so, because the usual integral over the lattice of all continuous functions on [0, 1] does not satisfy the condition (III). The author would have avoided the mistake had he been acquainted with the paper by H. Nakano, Proc. Imp. Acad. Tokyo **19** (1943), 138–143 [MR 8, 387], which treated the same problem about vector lattices. Applying this paper, we can correct the above point as follows: If I satisfies (I), (II) and $\sup_n I(y_n) \geq \inf_n I(z_n)$ for $L \ni y_n \uparrow y \in H$, $L \ni z_n \downarrow z \in H$, $y \geq z$, then we can extend I as described above. Finally, proposition 1 in § 1 is not correct; a counter-example can be easily found.} *H. Nakano* (Sapporo)

710:

Cristescu, Romulus. Sur quelques intégrales en lattices. Com. Acad. R. P. Romîne **8** (1958), 119–123. (Romanian. Russian and French summaries)

Eine Definition von O. Onicescu verallgemeinernd, erklärt der Verf. ein Integral für Funktionen und Maße mit Werten in einem Verband. Es seien $\mathfrak{E}$ ein Ring aus Teilmengen einer Menge E, L ein relativ vollständiger und unendlich distributiver Verband, φ eine additive Abbildung von $\mathfrak{E}$ in L und f eine beschränkte Abbildung von E in L. Das obere Integral von f über E hinsichtlich φ wird als untere Grenze aller $S(\Delta)$ definiert, wobei Δ alle endlichen Zerlegungen $\{A_i\}$ von E mit $A_i \in \mathfrak{E}$ durchläuft und $S(\Delta) = \mathbf{V}_i[(\mathbf{V}_{\xi \in A_i} f(\xi)) \wedge \varphi(A_i)]$ gesetzt ist; analog das untere Integral. Ohne Beweis werden eine Reihe von Eigenschaften dieser Integrale aufgezählt, unter anderem die Ordnungskonvergenz der oberen Integrale einer fallenden Folge von Integranden, die im Sinne des Verf. "in der oberen Grenze" konvergiert. Schließlich deutet der Verf. eine Ausdehnung der Definition auf den Fall nicht beschränkter Integranden an.

K. Krickeberg (Heidelberg)

711:

Král, Josef. On Lipschitzian mappings from the plane into Euclidean spaces. Czechoslovak Math. J. **8** (**83**) (1958), 257–266. (Russian summary)

Let K be an interval in the plane E_2, let $\Psi = (\Psi_1, \cdots, \Psi_r)$ be a Lipschitzian function on K whose range V lies in the Euclidean space E_r, and let $\Gamma = (\Gamma_1, \cdots, \Gamma_r)$ be a Lipschitzian function on V to E_r. Suppose M is a subset of V with inverse $\Psi^{-1}(M)$ of plane Lebesgue measure zero such that at each point $u^0 = (u_1^0, \cdots, u_r^0)$ of the set $V - M$ each Γ_i $(i = 1, \cdots, r)$ has a differential $\sum_{k=1}^r \tau_k^i(u^0)(u_k - u_k^0)$ relative to V. Insofar as the symbols involved are meaningful, set; for each $z \in K$,

$$\gamma(z) = \sum_{\substack{i,k=1 \\ i<k}}^{r} (\tau_i^k(\Psi(z)) - \tau_k^i(\Psi(z)))D_{ik}(z),$$

where $D_{ik}(z)$ is the Jacobian at $z = (x, y)$ of Ψ_i, Ψ_k with respect to x, y. As a result of several preliminary lemmas, it is shown that the function γ is defined a.e. in K and that

$$\int \Gamma(\Psi(z))d\Psi(z) = \int_K \gamma d\mu,$$

where μ is plane Lebesgue measure and the Stieltjes integral on the left is to be taken around the boundary of K in the positive sense.

T. A. Botts (Washington, D.C.)

FUNCTIONS OF A COMPLEX VARIABLE

712:

Parodi, Maurice. Une variante des méthodes de localisation des zéros d'un polynome. C. R. Acad. Sci. Paris **247** (1958), 669–670.

In this paper the polynomial

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n \quad (a_n \neq 0)$$

is written in a determinant form due to C. Bourlet, and this is transformed according to a method due to Chiò. Applying Hadamard's theorem to the latter determinant, the author concludes that the zeros of $f(z)$ lie in the union of the two disks

$$|z| \leq n-1, \quad (n-1)!|z+a_1| \leq (n-2)!|a_1| + \cdots + |a_n|.$$

M. Marden (Milwaukee, Wis.)

713:

Buck, R. C. Converse forms of the Hadamard product theorem. Duke Math. J. **26** (1959), 133–136.

Die Arbeit beschäftigt sich mit einer neuen und interessanten Umkehrung des Hadamardschen Produktsatzes. Es heisse $F \in \mathfrak{F}$, wenn für $F(z)$ ausser einem etwaigen Pol in ∞ alle singulären Stellen in einem abgeschlossenen Teil von $R(z) > 0$ liegen. Satz 1: Sind $F(z) = \sum a_n z^n$ und $G(z) = \sum b_n z^n$ beide in $\mathfrak{F}$, und ist $H(z) = \sum a_n b_n z^n$ ein Polynom, so ist $F(z)$ oder $G(z)$ ein Polynom. Eine rationale Produktfunktion $H(z)$ braucht jedoch keine rationalen Faktoren $F(z)$, $G(z)$ in $\mathfrak{F}$ zu haben. Satz 2 und Satz 2′ behandeln den Fall, dass $F(z)$, $G(z)$ aus $\mathfrak{F}$ sind und $H(z)$ an einer Stelle γ eine bestimmte Singularität (z.B. einen Pol) hat. Haupthilfsmittel beim Beweis ist der Satz des Verf. [dasselbe J. **13** (1946), 541–559; MR **8**, 371]: Liegen alle (endlichen und unendlichen) Singularitäten von $F(z) = \sum a_n z^n$ in $A^{-1} \leq |z| \leq A$, $|\arg z| \leq \alpha < \pi$, dann gibt es eine ganze Funktion $f(z)$ mit $f(n) = a_n$, für die

$$|f(z)| \leq O(1) \cdot \exp\{(A+\varepsilon)|x| + (\alpha+\varepsilon)|y|\} \quad (|z| \to \infty, \varepsilon > 0)$$

gilt. Satz 1 folgt daraus unmittelbar unter Verwendung des Satzes von Carlson. *D. Gaier* (Stuttgart)

714:

Komatu, Yûsaku. On convolution of power series. Kôdai Math. Sem. Rep. **10** (1958), 141–144.

Let $\mathfrak{R}$ represent the class of analytic functions which are regular and have positive real parts in the unit circle and which assume the value 1 at the origin. Let $\mathfrak{K}$ be the class of analytic functions which are regular in the unit circle and which map it univalently onto convex domains. Similarly $\mathfrak{St}$ denotes the class of analytic functions which are regular in the unit circle, which vanish at the origin, and which map the unit circle univalently onto regions star shaped with respect to the origin. The author proves the following theorems. (1) If

$$f(z) = 1 + 2\sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad g(z) = 1 + 2\sum_{n=1}^{\infty} b_n z^n$$

belong to $\mathfrak{R}$, then $h(z) = 1 + 2\sum_{n=1}^{\infty} a_n b_n z^n$ also belongs to $\mathfrak{R}$. (2) If f and g belong to $\mathfrak{K}$, and

$$\log f'(z) = 2\sum_{n=1}^{\infty} \frac{a_n}{n} z^n \quad \text{and} \quad \log g'(z) = 2\sum_{n=1}^{\infty} \frac{b_n}{n} z^n,$$

and if h is defined by

$$\log h'(z) = 2\sum_{n=1}^{\infty} \frac{a_n b_n}{n} z^n,$$

then h belongs to $\mathfrak{K}$. (3) If f and g belong to $\mathfrak{St}$, and if

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \frac{a_n}{n} z^n \quad \text{and} \quad \log \frac{g(z)}{z} = 2\sum_{n=1}^{\infty} \frac{b_n}{n} z^n,$$

and if h is defined by

$$\log \frac{h(z)}{z} = 2\sum_{n=1}^{\infty} \frac{a_n b_n}{n} z^n,$$

then h belongs to $\mathfrak{St}$. He also gives a similar theorem for normalized analytic functions in an annulus.

G. Springer (Lawrence, Kans.)

715:

Cugiani, Marco. Variazioni di segno condizionate, tratto ridotto e teorema di Fabry-Pólya. Riv. Mat. Univ. Parma **8** (1957), 99–132.

Let $f(z) = \sum a_n z^n$ in $|z| < 1$. The author proves theorems that assert the existence of a singular point $e^{i\theta}$ with $|\theta| \leq \pi\Delta$. Example: f has such a singular point provided there exist a sequence $\{n_h\}$ of integers ($n_h \to \infty$), a sequence $\{\beta_h\}$ of real numbers, and a positive constant θ satisfying the following three conditions.

(1) $$\Re a_{n_h} e^{i\beta_h} > 0, \quad (\Re a_{n_h} e^{i\beta_h})^{1/n_h} \to 1.$$

(2) If the interval I_h with midpoint n_h and length $2n_h^{7/8}\log^{4/3} n_h$ is covered with intervals

$$J_\nu = [(1 + n_h^{-3/8}\log n_h)^\nu n_h, (1 + n_h^{-3/8}\log n_h)^{\nu+1} n_h]$$

($\nu = \nu_h, \nu_h + 1, \cdots, 0, \cdots, \nu_h'$), then for each ν the number of variations of signs in the sequence $\{\Re a_m e^{i\beta_h}\}$ ($m \in J_\nu$) is at most $(\Delta + n_h^{-1/8}\log^{1/4} n_h) j_\nu$, where j_ν is the length of J_ν. (3) In the portion of the interval $[(1-\theta)n_h, (1+\theta)n_h]$ not covered by I_h,

$$|\Re a_m e^{i\beta_h}| \leq \exp\{2^{-1} n_h^{3/4}\log^{8/3} n_h\} \Re a_{n_h} e^{i\beta_h}.$$

G. Piranian (Ann Arbor, Mich.)

716:

Koosis, P. Proof of a theorem of the brothers Riesz. Studia Math. **17** (1958), 295–298.

The theorem of F. and M. Riesz, referred to in the title of the paper, states that if $F(\theta)$ is of period 2π and of bounded variation in $[-\pi, \pi]$ and if, with $z = re^{i\phi}$,

$$f(z) = \int_{-\pi}^{\pi} \frac{(1-r^2)dF(\theta)}{1 + r^2 - 2r\cos(\phi - \theta)}$$

is analytic in $|z| < 1$ then $F(\theta)$ is absolutely continuous. A proof of this theorem can be found in A. Zygmund, *Trigonometrical series* [Dover, New York, 1955; MR **17**, 361; § 7.53]. Another proof was given recently by H. Helson [Colloq. Math. **3** (1955), 113–117; MR **16**, 1016].

The author's proof is based on the following theorem [see S. Saks, *Theory of the Integral*, 2nd. ed., transl. by L. C. Young, Stechert, New York, 1937, Chapter IV, § 9]: $F(x)$ is absolutely continuous in $[a, b]$ if and only if $F'(x)$ exists and is finite everywhere in $[a, b]$, except on a point set M for which $\int_M |dF(x)| = 0$.

F. Herzog (East Lansing, Mich.)

717:

Chang, Ou-Yang. On an extremal problem in conformal mapping of septagons. Sci. Record (N.S.) **2** (1958), 71–75.

Als Siebeneck S sei bezeichnet ein einfach zusammenhängendes Gebiet, auf dessen Rand 7 erreichbare Randpunkte 1234567, in positivem Sinn durchlaufen, markiert sind. Es seien C_i ($i=1, 2, 3, 4$) die Klassen der lokal rektifizierbaren Jordankurven γ_i, die in S von 12 nach 34, von 34 nach 56, von 56 nach 12, und von 67 nach 12 verlaufen, und a_i seien vorgegebene Konstanten $\geqq 0$. Der Verfasser behandelt das Modulproblem für das Siebeneck S: Für nicht negative Funktionen $\rho(z) \in L_2(S)$, für die $\int_{\gamma_i} \rho(z)|dz| \geqq a_i$ ($\gamma_i \in C_i$; $i=1, 2, 3, 4$) gilt, wird die konforme Invariante $M(a_1, a_2, a_3, a_4) = \inf_\rho \iint_S \rho^2(z) db_z$ gesucht. In Anlehnung an die Untersuchungen von Jenkins für das 4-, 5-, und 6-Eck [Trans. Amer. Math. Soc. **67** (1949), 327–350; MR **11**, 341] wird bewiesen: (a) Zu jedem Quadrupel (a_1, a_2, a_3, a_4) existiert eine konforme Abbildung von S auf eines von 19 Normalgebieten; (b) hernach ist die extremale Metrik $\rho(z)=1$ f.ü. Die Existenz der extremalen Metrik ergibt sich auch aus einem allgemeinen Satz von Jenkins [Ann. of Math. (2) **66** (1957), 440–453; MR **19**, 845]. *D. Gaier* (Stuttgart)

718:

★Stoilow, S. Teoria funcţiilor de o variabilă complexă. Vol. 2. Funcţii armonice. Suprafeţe Riemanniene. [Theory of functions of a complex variable. Vol. 2. Harmonic functions. Riemann surfaces.] In collaboration with C. A. Cazacu. Editura Academiei Republicii Populare Romîne, Bucharest, 1958. 378 pp. 25.70 lei.

This is the second part of the author's lectures [cf. the review of Vol. I, 1954; MR **16**, 458]. Contents: The Dirichlet problem; Local properties of harmonic functions; The Dirichlet problem for multiply-connected domains; The Dirichlet integral and the minimum principle; Green's function, Lindelöf's principle, the principle of harmonic measure; Harmonic measure; Riemann surfaces; Analytic functions on closed Riemann surfaces; Analytic functions on open Riemann surfaces; Regularly and normally exhaustible Riemann surfaces.

R. P. Boas, Jr. (Evanston, Ill.)

719:

Heins, Maurice. Some constructive problems concerning analytische Gebilde. Math. Ann. **136** (1958), 9–22.

Recherche de solutions méromorphes de l'équation $b(b(z))=z$ (ou $b^2=I$). Préciser l'énoncé de ce problème conduit l'auteur à définir rigoureusement la notion d' "Analytische Gebilde" (traduction proposée: domaine d'uniformité) et à transformer le problème en celui de la recherche des domaines d'uniformité qui sont leurs propres inverses. S'il existe une involution τ conforme non triviale d'une surface de Riemann F, il existe sur F un domaine conformément équivalent à une couronne, invariant par τ et donnant une solution de $b^2=I$.

Les domaines d'uniformité sont introduits comme composantes du quotient de l'ensemble des couples ordonnés (f, g) (f et g fonctions méromorphes non constantes) par une relation d'équivalence naturelle. Cela conduit à la définition rigoureuse de la surface de Riemann d'une fonction analytique et à la notion de triplet "extendible": un triplet (F, f, g) (F surface de Riemann; f et g fonctions méromorphes sur F) est "extendible" si et seulement si il existe une surface de Riemann G et une application φ non univalente de F sur G telle que $f=f_1 \circ \varphi$ et $g=g_1 \circ \varphi$ (f_1 et g_1 sont méromorphes sur G): pour cette raison nous proposons le mot réductible pour exprimer la notion d' "extendibility". Cette même notion peut s'exprimer aussi: les ensembles des fonctions de la forme $f^{-1} \circ f$ et des fonctions de la forme $g^{-1} \circ g$ ont une intersection non réduite à l'identité.

La construction de la solution de $b^2=I$ permet de résoudre le problème de Riemann-Koebe: étant donnée une fonction m_1 uniforme sur une surface de Riemann F, déterminer une autre fonction m_2 uniforme sur F de telle sorte que le triplet (F, m_1, m_2) soit la surface de Riemann d'une fonction analytique (en fait $m_1 \circ m_2^{-1}$).

L. Fourès (Marseille)

720:

Cornea, A. Au sujet de parties singulières de la frontière idéale. Com. Acad. R. P. Romîne **8** (1958), 639–642. (Romanian. Russian and French summaries)

Making use of normal linear operators in boundary neighborhoods [L. Sario, Trans. Amer. Math. Soc. **72** (1952), 281–295; MR **13**, 735], the author introduces the concept of a singular part of type O_{HB} of the ideal boundary. He shows that a Riemann surface whose boundary contains such a singular part belongs to class O_{AB}. The result contains the theorem of Z. Kuramochi [Osaka Math. J. **6** (1954), 83–91; MR **16**, 233]: On a surface of class $O_{HB}-O_G$, the complement of any compact set belongs to O_{AB}. Corresponding results are established for classes O_{HD}, O_{AD}. *L. Sario* (Los Angeles, Calif.)

721:

Rankin, R. A. The Schwarzian derivative and uniformization. J. Analyse Math. **6** (1958), 149–167.

Soit R une surface de Riemann projettée sur le plan z. Soit $\hat{R}$ un revêtement de R simplement connexe, ramifié au dessus de certains points de R. t étant la variable d'uniformisation de R, $z(t)$ est automorphe pour un certain groupe homographique, pour lequel la dérivée de Schwarz $\{t, z\}$ est aussi automorphe. L'introduction de $\hat{R}$ conduit à un nouveau groupe comprenant des transformations elliptiques et paraboliques.

La troisième partie du mémoire est consacrée à l'étude de la forme de la dérivée $\{t, z\}$ suivant la nature du groupe linéaire d'automorphie d'une fonction $z(t)$.

L. Fourès (Marseille)

722:

Mulholland, H. P. Inequalities between the geometric mean difference and the polar moments of a plane distribution. J. London Math. Soc. **33** (1958), 260–270.

Démonstration de l'inégalité suivante: pour tout nombre réel $k>0$ et toute mesure $\mu \geqq 0$ du plan, de masse totale 1, on a

$$\exp\left[\iint \log|z-\zeta| d\mu(z) d\mu(\zeta)\right] \leqq (2/\sqrt{e})^{1/k}\left[\int |z|^k d\mu(z)\right]^{1/k}$$

l'égalité ayant lieu seulement lorsque μ est la mesure de densité $k|z|^{k-2}/4\pi m$ sur le disque $|z| \leqq (2m)^{1/k}$ ($m>0$).

La méthode repose sur les éléments de la théorie du potentiel logarithmique. Un résultat analogue avait été donné, à propos d'un problème de la théorie des nombres, par C. A. Rogers [Acta Math. **82** (1950), 185–208; MR **11**, 501] lorsque $k=1$ et que μ est portée par l'axe réel (la constante figurant au second membre est alors $\pi/2\sqrt{e}$). L'auteur annonce également des applications à la théorie des nombres. *J. Deny* (Strasbourg)

723:

Kövari, T. On theorems of G. Pólya and P. Turán. J. Analyse Math. **6** (1958), 323–332.

Sei $f(z)=\sum a_{\lambda_n}z^{\lambda_n}$ eine ganze Funktion und $M(r;\alpha,\beta)=\max_{\alpha\le\varphi\le\beta}|f(re^{i\varphi})|$, $M(r)=M(r;0,2\pi)$ gesetzt. Der Verfasser vergleicht das durch $M(r;\alpha,\beta)$ charakterisierte Anwachsen von $|f(z)|$ im Sektor $\alpha\le\arg z\le\beta$ mit dem Wachstum von $M(r)$ und präzisiert dabei Resultate von Pólya [Math. Z. **29** (1929), 549–640] und Turán [Rev. Math. Pures Appl. **1** (1956), no. 3, 27–32; MR **20** #6515]. Satz 1: Gilt $\lambda_n/n\to\infty$ und ist $f(z)$ von endlicher Ordnung, so ist für beliebiges $\delta>0$, $\varepsilon>0$

$$\log M(r;\alpha,\alpha+\delta)>(1-\varepsilon)\log M(r) \tag{1}$$

mit Ausnahme einer r-Menge der "logarithmischen Dichte 0". Satz 2: Gilt $\lambda_n/n\cdot\log^3\lambda_n>1$ und ist $f(z)$ von beliebiger Ordnung, so ist für jedes $\delta>0$, $\varepsilon>0$ (1) richtig mit Ausnahme einer r-Menge von 'endlichem logarithmischem Mass'. Bei Pólya und Turán war (1) jeweils nur für beliebig grosse r behauptet. Die Verschärfung gelingt durch eine geeignete Zerlegung $f(z)=\sum_{|\lambda_n-N|\le k}+\sum_{|\lambda_n-N|>k}$; hierin wird der erste Summand mit einer Ungleichung von Turán [loc. cit.] behandelt, der zweite hingegen mit Hilfssätzen von Clunie [J. London Math. Soc. **28** (1953), 58–66; **30** (1955), 32–42; MR **14**, 547; **16**, 684].

D. Gaier (Stuttgart)

724:

Kreyszig, Erwin; and Todd, John. The radius of univalence of the error function. Numer. Math. **1** (1959), 78–89.

The authors show that the radius of univalence ρ of

$$\operatorname{erf} z=\int_0^z\exp(-t^2)dt$$

is equal to the radius of the largest circle $|z|=r$ within which $\operatorname{erf} z$ is typically-real. The value of ρ lies in the interval $1.5746<\rho<1.5858$, and computations are given to indicate a plausible 7-decimal value of ρ to be 1.5748376.

The method of proof shows incidentally that $\operatorname{erf} z$ maps $|z|\le R=1.51\cdots$, where R is the largest positive root of the equation

$$(4R^4-1)^{1/2}-\arctan(4R^4-1)^{1/2}=\pi,$$

onto a schlicht domain convex in the direction of the real axis, but for no larger value of R. M. O. Reade [J. Math. Soc. Japan **9** (1957), 234–238; MR **19**, 642] showed that $\operatorname{erf} z$ was schlicht and close-to-convex in $|z|\le R=1.5\cdots$.

M. S. Robertson (New Brunswick, N.J.)

725:

Walsh, J. L. A generalization of Faber's polynomials. Math. Ann. **136** (1958), 23–33.

If E is the union of mutually disjoint, not one-point, continua $E_1,\cdots,E_K$ in the z-plane, each complement $-E_K$ being an infinite region, then there exist (up to translations uniquely) a univalent mapping with $z(\infty)=\infty$, $z'(\infty)=1$, of $-E$ onto a region $t(w)=\prod|w-a_k|^{m_k}t_0$, $m_k>0$, $\sum m_k=1$ [Walsh, Trans. Amer. Math. Soc. **82** (1956), 128–146; MR **18**, 290] and (nonuniquely) a sequence $a_{k_1},a_{k_2},\cdots$ with bounded $|nm_k-\sum_{j\le n,k_j=k}1|$ [Walsh, Ann. Mat. Pura Appl. (4) **39** (1955), 267–277; MR **17**, 1077]. For this mapping and sequence there exists (uniquely) an expansion, for $t(w_2)>t(w_1)>t_0$,

$$\frac{z'(w_2)}{z(w_2)-z(w_1)}=\sum_0^\infty\frac{b_n(z_n(w_1))}{c_{n+1}(w_2)},$$

$$c_n(w)=(w-a_{k_1})\cdot\cdots\cdot(w-a_{k_n}),$$

where

$$b_n(z)=(2\pi i)^{-1}\int_{t(w(Z))=T>t(w(z))}c_n(w(Z))dZ/(Z-z)$$

is a polynomial $z^n+\cdots$. Every function f analytic on E and for all z with $t_0<t(w(z))<t_1$ ($t_1\le\infty$) can be there expanded into $f=\sum d_nb_n(z)$ with

$$d_{n-1}=(2\pi i)^{-1}\int_{t(w)=t}f(z(w))dw/c_n(w),$$

and $s_n=\sum_0^n d_ib_i(z)$ converges fastest (geometrically) to f on E:

$$\limsup_{n\to\infty}(\max|f-s_n|\text{ on }E)^{1/n}=1/t_1.$$

Specializations: (1) For $K=1$, the b_n are the Faber polynomials of E. (2) For $z=w$, b_n becomes c_n, the series for f is a series of interpolation within a generalized lemniscate. (3) $z\equiv w$ and $K=1$, the Taylor series. (4) For $z\equiv w$ and rational m_k, an amplified Jacobi series.

A cornerstone of the proofs is the lemma: c_nt^{-n} and its reciprocal are bounded on every closed set containing no a_k.

T. S. Motzkin (Los Angeles, Calif.)

726:

Riekstynya, V. Ž. A generalization of asymptotic expansions. Latvijas Valsts Univ. Zinātn. Raksti **20** (1958), no. 3, 145–152. (Russian. Latvian summary)

An approximation which is essentially of an asymptotic nature may satisfy a condition such as $|F(z)-f(z)|<\varepsilon|f(z)|$ except near the zeros of $F(z)$. Definitions are developed which permit the discussion of asymptotic expansions of this character.

$f(z)$ and $F(z)$ are related by a Laplace integral of the form $F(t)=(2\pi i)^{-1}\int e^{zt}f(z)dz$. The contour of integration extends to infinity along radii $\arg(z-z_0)=\pm\Phi(\frac{1}{2}\pi<\Phi<\pi)$. The singular point z_0 of $f(z)$ is to the left of the contour. Then from an asymptotic expansion of $f(z)$ near z_0 in the form $\exp\{-c(z-z_0)^{-\mu}\}\sum a_n(z-z_0)^{\lambda_n}$ is derived an asymptotic expansion of the generalized kind for $F(t)$ and large t which proceeds in terms of generalized Bessel functions.

This discussion is extended by considering several singular points of $f(z)$ and asymptotic expansion of $f(z)$ depending on powers of $\log(z-z_0)$.

A. J. Macintyre (Cincinnati, Ohio)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

See also 889.

727:

Bremermann, Hans J. Die Charakterisierung Rungescher Gebiete durch plurisubharmonische Funktionen. Math. Ann. **136** (1958), 173–186.

A pair (D,D') of domains of holomorphy in C^n is called a Runge pair if D is contained in D' and if the set of all

holomorphic functions on D' is dense in the set of all holomorphic functions on D, equipped with the topology of compact convergence. The main result of this paper is: (D, D') is a Runge pair if and only if $D=\lim D_k$, $D_1 \subseteq D_2 \subseteq \cdots \subseteq D$, where $D_k=\{z: V_k(z)<0\}$ and $V_k(z)$ is plurisubharmonic in D'. $W(z)$ is called plurisubharmonic in D' if for every relatively compact subset D_0 of D' there is a sequence $W_1, W_2, \cdots$ of C^∞-functions on D_0 such that $W_1(z) \geqq W_2(z) \geqq \cdots \geqq W(z)$, $W(z)=\lim W_k(z)$, and

$$\sum_1^n \frac{\partial^2 W_k(z)}{\partial z_i \partial \bar{z}_j}\, dz_i d\bar{z}_j$$

is positive semidefinite. Consequences: (i) If (D, D') is a Runge pair and E a (1-dimensional) complex line, then $D \cap E$ is relatively simply connected with respect to $D' \cap E$. (ii) If D is a domain of holomorphy which is convex (in the usual sense), then (D, C^n) is a Runge pair. (iii) Let $D=\{z: V(z)<0\}$ where $V(z)$ is plurisubharmonic in some neighborhood of $\bar{D}$; then the Banach algebra of all continuous functions on $\bar{D}$ which are holomorphic in D is dense in the algebra of all holomorphic functions on D, equipped with the topology of uniform convergence.

H. Roehrl (Chicago, Ill.)

728:

Pyateckiĭ-Šapiro, I. I. On a problem proposed by E. Cartan. Dokl. Akad. Nauk SSSR **124** (1959), 272–273. (Russian)

This note provides an example of a bounded homogeneous domain in the space of five complex variables which is not symmetric, thus answering negatively a question of E. Cartan. Let $Z=\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$ and $U=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, and let H be the set of all points (Z, U) for which the matrix $-i(Z-\bar{Z})-U\bar{U}'-\bar{U}U'$ is positive definite. The author proves that H is analytically equivalent to a bounded domain, and that H is homogenous but not symmetric. A second example in four variables is given without proof.

F. D. Quigley (New Orleans, La.)

729:

Stoll, Wilhelm. Über die Fortsetzbarkeit analytischer Mengen endlichen Oberflächeninhaltes. Arch. Math. **9** (1958), 167–175.

Sur une variété analytique complexe G, de dimension complexe n, considérons un ensemble analytique S de dimension q; soit N un ensemble analytique défini dans $G-S$, et de dimension homogène p $(q<n, p<n)$. Le problème de prolonger N de $G-S$ à G donne lieu aux résultats suivants.

(1°) Si $q<p$, l'adhérence $\bar{N}$ de N constitue toujours le prolongement cherché [cf. R. Remmert et K. Stein, Math. Ann. **126** (1953), 263–308; MR **15**, 615].

(2°) Si $q \geqq p$, l'analyse est plus délicate. Si $q=p$, on sait que N est prolongeable dès que chaque composante irréductible de S contient un point régulier pour $\bar{N}$. Dans la recherche de conditions métriques, un résultat précédent de l'auteur obtenu pour $p=q=n-1$ peut s'énoncer, si l'on fait usage de la notion d'aire [cf. P. Lelong, Bull. Soc. Math. France **85** (1957), 239–262; MR **20** #2465]: il suffit que N ait une aire bornée pour que $\bar{N}$ prolonge N dans G. Ce résultat est ici étendu au cas $p=q<n$, par l'énoncé: si il existe une forme hermitienne définie positive dont l'intégrale sur N est bornée, N se prolonge par $\bar{N}$ dans G. Le mémoire contient de plus la conjecture, étayée par l'examen d'un cas particulier, que l'énoncé demeure vrai pour $q>p$.

P. Lelong (Paris)

730:

Sommer, Friedrich. Komplex-analytische Blätterung reeller Mannigfaltigkeiten im C^n. Math. Ann. **136** (1958), 111–133.

Let C^n denote the n-dimensional vector space over the complex numbers of all n-tuples $x=(x_1, \cdots, x_n)$ where $x_j=x_j'+ix_j''$ are complex numbers. This space can be considered simultaneously as a $2n$-dimensional vector space over the real numbers and as such it has two automorphisms: $x \to ix$ and $x \to \bar{x}$. The author first considers real planes (not necessarily through the origin) given, for example, as the image of a real vector space $V_R{}^m$ under a linear map: $x=Lt+x_0$, $t \in V_R{}^m$. He shows that such a manifold is a family, with $2l-r$ real parameters, of complex planes of dimension $r-l$ if and only if rank $L=l$ and rank $\begin{pmatrix} L \\ \bar{L} \end{pmatrix}=r$, the rank being with respect to the real numbers. A similar criterion is given for the linear manifold of solutions of a system of equations $Bx+\bar{B}\bar{x}+y=0$. These criteria are then applied to the tangent vectors to a submanifold of C^n given by the vanishing of real functions in x_i', x_i'' (or equivalently in x_i, $\bar{x}_i$). The main result is the following: A submanifold M^{2n-r} of C^n which is of class C^2 is a $2l-r$ parameter family of $n-l$ dimensional analytic submanifolds $M_{C^{n-l}}$ in the neighborhood of a point P if and only if at each point in this neighborhood the tangent space to $M_{C^{n-l}}$ contains an $n-l$ dimensional complex subspace (spanned by $n-l$ vectors of type $(1, 0)$) such that the field of subspaces together with their images under complex conjugation forms an integrable distribution.

This criterion is then applied to several special cases; in particular, a simple condition is given for a hypersurface M^{2n-1} in C^n to be a one parameter family of analytic manifolds.

W. M. Boothby (Evanston, Ill.)

731:

Togari, Yoshio. On ramified Riemann domains. Nagoya Math. J. **14** (1959), 173–191.

A Riemann domain is defined to be a pair (E, f) consisting of a n-dimensional normal complex space E and a holomorphic mapping f of E into C^n such that for every element x of E the set $f^{-1}(f(x))$ is discrete in E. If (E, f) is a Riemann domain, then E is proved to satisfy the second countability axiom. The boundary dE of E consists of those filters r satisfying the following conditions: (1) r has no accumulation point in E; (2) $f(r)$ converges to some point z of C^n; (3) r has a base consisting of connected components of $f^{-1}(U)$, U being some neighborhood of z; (4) given any neighborhood U of z, one of the connected components of $f^{-1}(U)$ belongs to the base. Then $E \cup dE$ also satisfies the second countability axiom. Generalising a well-known result the author shows that for every Riemann domain (E, f) and every function F holomorphic on (E, f) there is a maximal Riemann domain containing (E, f) to which F may be prolonged holomorphically. The main theorem of the paper is: If a Riemann domain (E, f) has the properties (i) for every pair x, y of elements of E $(x \neq y, f(x)=f(y))$ there is a holomorphic function on E whose terms in x and y are different, and (ii) for every compact subset K of E and every boundary point r, the

holomorphic envelope of K does not belong to r; then (E, f) is a domain of holomorphy. A pair (E, f), (E', f') of Riemann domains is called a Runge pair if (a) E is an open subset of E' and the restriction of f' onto E coincides with f, and (b) the set of all holomorphic functions on E' is dense in the set of all holomorphic functions on E, equipped with the topology of compact convergence. The author proves: If (E_k, f_k) is a sequence of Riemann domains satisfying (i) and (ii), and if every pair of successive domains is a Runge pair, then the limit domain is also a Riemann domain satisfying (i) and (ii). *H. Roehrl* (Chicago, Ill.)

732:

Kajiwara, Joji. On the equivalence of Hitotumatu's conjecture and the decomposition theorem. Mem. Fac. Sci. Kyusyu Univ. Ser. A **12** (1958), 113–135.

The author develops a relative cohomology theory of pairs (X, A) (consisting of a complex space X and an open subset A of X) with coefficients in an analytic sheaf. All the axioms of a cohomology theory are fulfilled with the exception of the excision axiom; the latter has to be replaced by: let F be an analytic sheaf and let D_1, D_2 be relatively compact open subsets of X such that

$$D_1 \cap D_2 \neq \emptyset \quad \text{and} \quad (\overline{D_1 - D_1 \cap D_2}) \cap (\overline{D_2 - D_1 \cap D_2}) = \emptyset;$$

then the inclusion mapping $j\colon (D_1, D_1 \cap D_2) \to (D_1 \cup D_2, D_2)$ induces isomorphisms

$$j^*\colon H^q(D_1 \cup D_2, D_2; F) \to H^q(D_1, D_1 \cap D_2; F) \text{ for all } q \geq 0.$$

The usual algebraic procedure leads to the Mayer-Vietoris cohomology sequence with coefficients in F. {It may be observed that actually all of the above cohomology theory developed by the author is valid word for word for pairs (X, A), X being a ringed space and A an open subset of X, with coefficients in a sheaf of modules over the structure sheaf.}

From the above facts one concludes easily the following. (i) If $D_1 \cup D_2$ is holomorphically complete and s a section in F over $D_1 \cap D_2$, then there are sections s_1 and s_2 in F over D_1 and D_2 respectively such that over $D_1 \cap D_2$ the equation $s_1 - s_2 = s$ holds. If the property (i) is fulfilled with F being the sheaf $\mathcal{O}$ of germs of holomorphic functions, then we say that $D_1 \cup D_2$ is functionally decomposed into D_1 and D_2. (ii) Under the above hypothesis the following two statements are equivalent: (a) $H^1(D_1 \cup D_2, \mathcal{O}) = (0)$; (b) $D_1 \cup D_2$ is functionally decomposed into D_1 and D_2.
H. Roehrl (Chicago, Ill.)

SPECIAL FUNCTIONS

See also 741, 969.

733:

Ragab, F. M. Series of products of Bessel polynomials. Canad. J. Math. **11** (1959), 156–160.

Some finite series of the type $\sum_0^n C_r \gamma_r(x, a, b)\gamma_r(y, a, b)$ are summed in terms of the nth Bessel polynomial $\gamma_n(x, a, b)$. *N. D. Kazarinoff* (Ann Arbor, Mich.)

734:

★**Robin, Louis. Fonctions sphériques de Legendre et fonctions sphéroïdales. Tome II.** Gauthier-Villars, Paris, 1958. viii + 384 pp. Broché 5000 francs; cartonné 5300 francs.

[Pour l'analyse du Tome I (1957) voir MR **19**, 954.]

Ce volume qui comprend 3 chapitres débute par une magistrale étude des "Fonctions associées de Legendre de type général", $P_n^m(\mu)$ et $Q_n^m(\mu)$, où les indices n et m ne sont plus entiers comme dans le tome I, mais quelconques, réels ou complexes (Ch. IV, pp. 1 à 220).

L'auteur part de l'équation différentielle de Legendre associée, exprime P_n^m et Q_n^m par la fonction hypergéométrique, puis sous forme d'intégrales de contour. Il en déduit les relations linéaires et homogènes entre P_n^m, Q_n^m, P_n^{-m}, Q_{-n-1}^m et donne de ces fonctions des développements en séries de puissance de

$$\frac{1}{1-\mu^2},\ \frac{2}{1-\mu},\ \frac{2}{1+\mu},\ \frac{\mu-1}{\mu+1},\ 1-\mu^2,\ \frac{\mu^2-1}{\mu^2},$$

puis recherche encore d'autres séries entières rapidement convergentes au voisinage des points ± 1. Cette recherche, très intéressante, mais qui donne lieu à de très longs calculs, se termine par des considérations qui retiennent l'attention sur les singularités de $P_n^m(\mu)$ et $Q_n^m(\mu)$, envisagées comme des fonctions analytiques de μ, n et m. D'autres formules (très compliquées) sont données ensuite fournissant les développements en puissances de $(\mu - \sqrt{(\mu^2-1)})^2$, où l'intérêt de ces longs calculs n'est qu'insuffisamment dégagée.

On revient ensuite à la représentation de P_n^m et Q_n^m par des intégrales définies prises le long de segments de l'axe réel d'intégrales du type Laplace-Heine (classiques si m et n sont entiers, et étendues au cas actuel de m et n quelconques par Erdélyi); et l'on donne une généralisation des intégrales de Dirichlet et de Mehler (appliquées habituellement aux polynomes de Legendre), suivie de représentations intégrales de $P_n^m(\operatorname{ch}\psi)$ et $Q_n^m(\operatorname{ch}\psi)$ par la formule de Whipple. On établit ensuite les relations de récurrence, vérifiées pour P_n^m et Q_n^m pour les valeurs de n et de m successives (différant de une ou 2 unités: ces relations sont les mêmes que pour m et n entiers). Ces relations, dont les coefficients sont analytiques, sont indéfiniment dérivables.

Une longue étude est consacrée aux dérivées de $P_n^m(\mu)$ et $Q_n^m(\mu)$ par rapport à μ, n et m. On obtient, en particulier, des formules donnant $(\partial P_n^m/\partial n)_{n=N}$, n étant entier ou valant la moitié d'un impair. A cette occasion on précise ce que deviennent les formules précédentes dans des cas particuliers simples: $n=0$, $m=\pm n$.

L'auteur signale et traite ici des applications des considérations précédentes au calcul d'intégrales définies renfermant des fonctions de Legendre telles que

$$\int_0^1 \mu^l P_n(\mu)\,d\mu, \quad \int_0^1 [P_n^m(\mu)]^2 d\mu,$$

$$\int_0^1 [Q_n^m(\mu)]^2 d\mu, \quad \int_0^1 P_n^m(\mu) Q_n^m(\mu)\,d\mu.$$

Il donne enfin d'expression de P_n^m et Q_n^m au moyen d'intégrales contenant des fonctions de Bessel (et inversement l'expression des fonctions de Bessel au moyen d'intégrales contenant les fonctions de Legendre) par des formules telles que

$$P_n^{-m}(\cos\theta) = \frac{1}{(n+m)!}\int_0^\infty t^n e^{-t\cos\theta} J_m'(t\sin\theta)\,dt$$

(où $R(u+m) > -1$, $0 < \theta < \frac{1}{2}\pi$),

et

$$J_{n+\frac{1}{2}}(z) = (2z/\pi)^{\frac{1}{2}} \int_0^1 \cos(\mu z - n\pi/2) P_n(\mu)\,d\mu.$$

Le chapitre V, beaucoup plus court (pp. 221–278), est consacré aux valeurs approchées, et essentiellement aux développements asymptotiques de P_n^m et Q_n^m, tout d'abord pour $|n|$ grand, m et $n=\mu$ étant fixes (avec discussions pour les différentes régions du plan complexe). Puis m est supposé grand à son tour (problème plus simple); enfin m et n sont simultanément grands (travaux de H. Jeffreys et de B. H. Jeffreys). L'étude est complétée par un aperçu historique de la résolution de ces questions avec une substantielle bibliographie.

On passe ensuite aux inégalités vérifiées par P_n^m et Q_n^m (résultats en majorité dûs à Hobson), comprenant celles toutes récentes obtenues pour les polynomes de Legendre

$$|P_n^{(k)}(\mu)| < \frac{(2n-1)!!}{(n-k)!}(|\mu|+1)^{n-k} \quad \text{(Picone, 1953)}$$

et

$$[P_n(\mu)]^2 - P_{n-1}(\mu)P_{n+1}(\mu) \quad \text{(Turán, 1955)}.$$

Le chapitre VI traite des séries de polynomes de Legendre et en particulier des développements d'une fonction: théorème d'équiconvergence, de la sommabilité Abel-Poisson, étude des moyennes de Césaro, théorème de Szegö sur la sommabilité (C, k) aux points ± 1 (avec des exemples de fonctions continues dont la série de Legendre diverge en ces points). Suivent des exemples en nombre considérable de développements en séries de Legendre (avec un exemple de développement en séries dont l'élément est $P_{n_p}^m$, les n_p n'étant pas entiers, mais déterminés par un équation transcendante, développement en rapport avec un problème aux limites relatif à l'équation de Laplace ou à celle des vibrations).

La fin du volume est consacrée aux séries de polynomes de Laplace. Débutant par un historique substantiel de la question, l'auteur étudie de nouveau ici la convergence, la sommabilité Abel-Poisson-Cesàro, et enfin le phénomène de Gibbs dans les séries de Legendre et de Laplace.

Ce second volume, d'un contenu extrêmement riche, apporte plus d'éléments nouveaux encore que le premier, les questions qu'il traite étant moins classiques (les fonctions qu'il se propose d'étudier étant elles-mêmes moins utilisées).

Dans cet ouvrage exceptionnellement substantiel, le côté qui laisse le plus à désirer est sans doute celui de l'exposition. Les calculs sont très bien expliqués, très détaillés, on n'oublie rien mais on ne sait pas toujours bien où l'on veut en venir. On aimerait que l'auteur nous renseigne non seulement sur l'historique de la question, ce qu'il fait en général avec scrupule, mais aussi sur le projet qu'elle se propose. Les calculs nous sont apportés les uns après les autres, mais gardent un air juxtaposé comme les termes d'une addition qui finalement on n'effectue pas. Dans le domaine de l'information, il est inconcevable qu'on puisse mieux faire; le livre est plus qu'un traité, c'est une encyclopédie. *R. Campbell* (Caen)

735:

Ragab, F. M. An integral involving the associated Legendre functions of the first kind. Michigan Math. J. **6** (1959), 97–99.

If $\Re(l-m+n)>0$, $\Re(l-m-n)>1$ and $\Re m>-1$, then

$$\int_0^\infty (\sinh u)^{m+1} P_n^{-m}(\cosh u)(\cosh u)^{-l} E(p;\alpha_r:q;\rho_s:z\cosh^2 u)\,du = 2^{-m-1} E\left[\begin{matrix}\alpha_1, \cdots, \alpha_p, \frac{1}{2}(l-m+n), \frac{1}{2}(l-m-n-1) \\ \rho_1, \cdots, \rho_s, \frac{1}{2}l, \frac{1}{2}(l+1)\end{matrix}; z\right].$$

This result provides a description of the integral on the left when $|z|\to\infty$. *N. D. Kazarinoff* (Ann Arbor, Mich.)

736:

Peyser, Gideon. Note on the derivatives of the Legendre polynomials. Math. Mag. **31** (1957/58), 210.

On $[-1, 1]$, $\max_{[-1,1]} d^m P_n(x)/dx^m = d^m P_n(x)/dx^m$ if and only if $x=1$. *N. D. Kazarinoff* (Ann Arbor, Mich.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 903, 991, 992, 1033.

737:

Volpato, Mario. Sulla derivabilità, rispetto a valori iniziali ed a parametri, delle soluzioni dei sistemi di equazioni differenziali ordinarie del primo ordine. Rend. Sem. Mat. Univ. Padova **28** (1958), 71–106.

Consider $dy/dx=f(x, y, \lambda)$, where y, f are n-vectors, x, λ scalars, f defined in $a\leqq x\leqq b$, $(y, \lambda)\in R^{n+1}$; it is assumed (Carathéodory) that f is summable for fixed (y, λ) and

$$\|f(x, y, \lambda)-f(x, y', \lambda')\| \leqq P(x)(\|y-y'\|+|\lambda-\lambda'|)$$

with a summable P; the last assumption ensures the existence almost everywhere of the derivatives of f with respect to y, λ. $y=\varphi(x, \xi, \eta, \lambda)$ being the (unique) general solution, $\varphi(\xi, \xi, \eta, \lambda)=\eta$, it is shown that the derivatives of φ with respect to ξ, η, λ exist for all x on almost all lines $(\xi, \eta, \lambda)=$const and that these derivatives (and the Jacobians) satisfy the classical equations and relations. If moreover f is continuous and has continuous partials with respect to y for each $x=$const, the derivatives of φ with respect to η are continuous and φ_ξ is continuous for almost all ξ. Conditions for the existence of second order derivatives are also considered. *J. L. Massera* (Montevideo)

738:

Baiada, E.; e Lorefice, M. I metodi di approssimazione nello studio dei sistemi differenziali ordinari. Atti Accad. Sci. Lett. Arti Palermo. Parte I (4) **16** (1955/56), 193–223 (1957).

This paper is devoted to a real differential system

$$(*) \qquad y_i' = f_i[x; y_1(x), \cdots, y_m(x)], \quad y_i(x_0) = y_1^{(0)} \quad (i = 1, \cdots, m),$$

such that on the $(m+1)$-dimensional interval R: $|x-x_0|\leqq a$, $|y_i-y_i^{(0)}|\leqq b$ $(i=1, \cdots, m)$ the functions f_i satisfy the conditions of Carathéodory, that is, each f_i is continuous in $[y_1, \cdots, y_m]$ for fixed x, and measurable in x for fixed $[y_1, \cdots, y_m]$, while there is a function $M(x)$ which is Lebesgue integrable on $|x-x_0|\leqq a$ such that

$$|f_i[x; y_1, \cdots, y_m]| \leqq M(x) \quad (i = 1, \cdots, m)$$

on R. The authors are concerned with approximation sequences $\{y_i^{(n)}(x)\}$ $(n=1, 2, \cdots)$ of the form

$$(**) \quad y_i^{(n)}(x) = y_i^{(0)} + \sigma_i(x, d/n), \quad x_0 \leqq x \leqq x_0+d/n,$$

$$y_i^{(n)}(x) = y_i^{(0)} + \sigma_i(x, d/n) + \varepsilon_i(x, d/n) + \int_{x_0}^{x-d/n} f_i[t; y_1^{(n)}(t), \cdots, y_m^{(n)}(t)]\,dt, \quad x_0+d/n \leqq x \leqq x_0+d.$$

Here d is a positive constant satisfying $0<d\leq a$ and $2\int_{x_0}^{x_0+d} M(t)dt\leq b$; while $\sigma_i(x, d/n)$, $\varepsilon_i(x, d/n)$ $(i=1, \cdots, m;$ $n=1, 2, \cdots;$ $x_0\leq x\leq x_0+d)$ are such that (i) $\varepsilon_i(x_0+d/n, d/n)=0$, (ii) on $x_0\leq x\leq x_0+d$ the functions ε_i and σ_i are continuous in x, uniformly with respect to n, (iii) $\sigma_i(x, d/n)\equiv\sigma_i(x_0+d/n, d/n)$ on $x_0+d/n\leq x\leq x_0+d$, (iv) $\varepsilon_i\to 0$ and $\sigma_i\to 0$ as $n\to\infty$, uniformly for x on $x_0\leq x\leq x_0+d$. In the first part of the paper it is shown that the functions of such a sequence (**) are equibounded and equi-continuous on $x_0\leq x\leq x_0+d$, so that any $(y_i(x))$ which is the uniform limit on this interval of a subsequence of (**) is a solution of (*). Conversely, for each solution $(y_i(x))$ of (*) there are functions ε_i, σ_i satisfying the conditions (i)–(iv), and such that $y_i^{(n)}(x)\to y_i(x)$ uniformly on $x_0\leq x\leq x_0+d$; indeed, by proper choice of ε_i, σ_i the author obtains $y_i^{(n)}(x)\equiv y_i(x)$ $(n=1, 2, \cdots)$. The second portion of the paper applies the above results in the proof of the existence of solutions of (*) of a particular type, termed "pseudo-extreme", and which is a generalization to systems (*) of the upper and lower solutions for the case $m=1$. In particular, whenever the f_i possess certain monotoneity properties this latter result provides a generalization of a result of T. Ważewski [Ann. Soc. Polon. Math. **16** (1938), 97–111].

W. T. Reid (Evanston, Ill.)

739a:

Denjoy, Arnaud. Le phénomène ergodique et les trajectoires sur le tore. C. R. Acad. Sci. Paris **247** (1958), 1072–1078.

739b:

Denjoy, Arnaud. Les systèmes d'équations différentielles périodiques. C. R. Acad. Sci. Paris **247** (1958), 1691–1696.

739c:

Denjoy, Arnaud. Équations différentielles périodiques. C. R. Acad. Sci. Paris **247** (1958), 1923–1928.

739d:

Denjoy, Arnaud. Sur les équations différentielles périodiques. C. R. Acad. Sci. Paris **248** (1959), 28–33.

739e:

Denjoy, Arnaud. Les équations différentielles périodiques. Allure asymptotique des intégrales. C. R. Acad. Sci. Paris **248** (1959), 325–330.

The first of these papers contains some remarks on the functions arising in the author's classical treatment of vector fields on the torus [J. Math. Pures Appl. (9) **11** (1932), 333–375], including conditions for various pseudo-periodicities. In the succeeding papers, after mentioning some complements to the first, a beginning is made to the study of differential equations on the n-torus. The case $n=3$ is singled out and examined by means of a surface of section. By describing the integral curves, constructions of examples of great complexity are indicated. Many interesting problems are suggested.

L. W. Green (Minneapolis, Minn.)

740:

Hsu, C. S. On simple subharmonics. Quart. Appl. Math. **17** (1959), 102–105.

A solution of the equation (1) $x''+f(x)=P_0\cos\omega t$ is called a simple subharmonic of order $(1/r)$ if it has the form $x=x_0\cos(\omega t/r)$. The author applies the transformation (*) $\xi=T_r(\eta)$, where T_r is the Tchebycheff polynomial of order r, to the equation (3) $\xi''+(1+k)\xi=k\cos t$, and obtains that (4) $\eta''+r^{-2}\eta+kT_r(\eta)=k\cos t$ has the simple subharmonic solution $\eta=\cos(t/r)$. This result has been obtained by R. M. Rosenberg [same Quart. **15** (1958), 341–354; MR **19**, 1053]. In the same way it is shown that if r is even

$$\xi''+r^{-2}\xi+k(-1)^{r/2}T_r(\xi)=k\cos t$$

has the solution $\xi=\sin(t/r)$, and that if r is odd,

$$\xi''+r^{-2}\xi+k(-1)^{(r-1)/2}T_r(\xi)=k\sin t$$

has the solution $\xi=\sin(t/r)$.

The author then deduces that the simple subharmonics found are a unique steady state for the nonlinear equations. The argument is not valid since the transformation (*) does not transform equation (3) into equation (4), except for the solutions $\xi=\cos t$, $\eta=\cos(t/r)$.

W. S. Loud (Minneapolis, Minn.)

741:

★Patry, Jean. Über die linearen Differentialgleichungen mit sinusförmigen Koeffizienten. Promotionsarbeit. Eidgenössische Technische Hochschule, Zürich, 1957. 151 pp.

L'auteur fait le point des résultats obtenus relativement à l'intégration de l'équation différentielle linéaire du second ordre à coefficients périodiques, signale le caractère difficile des calculs auxquels elle conduit et l'imperfection théorique qui laisse toujours, dans ces questions, le lecteur sur sa soif.

Dans une première partie, il étudie les méthodes générales d'intégration. Il les ramène (selon un procédé approfondi par Lindsay, Ince, Goldstein, Whittaker) à la résolution d'une équation linéaire aux différences finies à 3 termes (se prêtant bien aux calculs numériques à la machine). La mise en train de la méthode et la recherche des singularités de l'équation sont fondées sur les théorèmes de Fuchs et de Floquet.

Les procédés employés sont mis en rapport avec ceux de Hill et la théorie des fractions continues, et les méthodes de calcul illustrées par des graphiques. Des tables numériques ont été dressées, des applications sont données à des problèmes de physique avec conditions aux limites (électrostatique).

Dans une seconde partie sont étudiées les équations qui ne subissent que des modifications simples par le changement de x en $-x$. On s'attache particulièrement à la détermination des "Doppel-lösungen", c'est-à-dire des cas où les 2 solutions fondamentales de l'équation ont des propriétés analogues (par exemple: être périodiques); on sait que de tels cas sont possibles pour les solutions de période 2π de l'équation qu'il appelle "de Mathieu inverse" et qui s'écrit:

$$(1+2\theta\cos x)u''+cu=0.$$

Les équations admettant des "Doppel-lösungen" ont d'autres propriétés, qui sont aussi étudiées (la forme des solutions, leurs symétries ou antisymétries, la forme des déterminants caractéristiques).

Un exemple d'application pratique de la méthode numérique est donné, celui des réseaux en optique. On

étudie de nouveau à ce propos les solutions de l'équation de Mathieu et d'une forme généralisée de son inverse

$$y'' + \left[\frac{k^2}{a+b\cos x} - l^2\right] y = 0.$$

On étudie également une autre équation admettant des "Doppel-lösungen", celle de Whittaker:

$$u'' + (a + b\cos 2x + c\cos 4x)u = 0.$$

Des graphiques sont donnés [déjà mis en évidence par Ince dans un article ancien (1923)], ainsi que pour les valeurs propres de l'équation de Mathieu. L'auteur applique enfin sa méthode à une équation du 3ème ordre

$$u''' + 2a\sin x \cdot u = 0.$$

La 3ème partie de l'ouvrage, la plus courte (pp. 108–131) est consacrée aux équations non homogènes, et tout d'abord à

$$y' + (A - e^{-x})y = A.$$

Un retour est opéré, à ce propos, sur la détermination des fonctions de Mathieu de 2ème espèce.

L'ouvrage, très documenté, n'apporte de nouveauté que dans l'exposition et l'application des méthodes numériques; il y a peu de résultats nouveaux, mais il est précieux par ses mises au point, et sa clarté.

Une bibliographie le termine, très minutieusement et très complètement élaborée. *R. Campbell* (Caen)

742:

Patry, Jean. Sur la résolution numérique dans les cas limites des équations différentielles linéaires à coefficients sinusoïdaux. Z. Angew. Math. Phys. **10** (1959), 35–72.

Under the transformation $Z = \exp(ix)$ the equation

$$\sum_{k=0}^{s} (e_k + f_k e^{-ix} + g_k e^{ix}) \frac{d^k u}{dx^k} = 0$$

goes into an equation of the form

$$\sum_{k}^{s} (F_k + E_k Z + G_k Z^2) Z^k \frac{d^k u}{dZ^k} = 0.$$

The latter equation has four singular points: $Z = 0, \infty, Z_1, Z_2$, where $|Z_1| \leqq |Z_2|$. Assuming that $|Z_1| < |Z_2|$, the author has previously used generalizations of the methods used in the case of the Mathieu equation to obtain solutions of the form $u = \sum_{n=-\infty}^{\infty} D_n Z^{\mu+n}$ in the three domains $|Z| < |Z_1|$, $|Z_1| < |Z| < |Z_2|$, $|Z| > |Z_2|$. [C. R. Séances Soc. Phys. Hist. Nat. Genève **59** (1942), 122–126; MR **7**, 204. Also a book, #741 above.] In the present paper it is shown that these methods are also applicable, under broad conditions, when Z lies on the common boundary of two of the domains, and this even when $|Z_1| = |Z_2|$ so that the second domain disappears. The computational procedures used in the general case are illustrated and tested extensively by being applied to the case of an elementary equation which can be solved by quadratures.

L. A. MacColl (New York, N.Y.)

743:

Heinz, Erhard. Halbbeschränktheit gewöhnlicher Differentialoperatoren höherer Ordnung. Vorwort von B. L. van der Waerden. Math. Ann. **135** (1958), 1–49.

In his lecture at the Amsterdam Congress [Proc. Int. Congr. Math., 1954, Amsterdam, vol. III, pp. 243–250, Noordhoff, Groningen, 1956; MR **19**, 550], F. Rellich announced two criteria for the semiboundedness of a singular, ordinary differential operator but left proofs and some further results to a subsequent paper. The latter paper was unfinished, however, at Rellich's death. The task of completing it was undertaken by E. Heinz, and it is the outcome of these labors which appears in the work under review. Let

$$Du \equiv (k(x))^{-1} \sum_{\nu=0}^{m} (-1)^{m-\nu} (p_\nu(x) u^{(m-\nu)})^{(m-\nu)}$$

define a formally self-adjoint ordinary differential operator with real, sufficiently differentiable coefficients such that $p_0 > 0$, $k > 0$ on the interval $a < x < b$ $(-\infty \leqq a \leqq b < \infty)$. Let $\mathfrak{P}$ be the space of functions with continuous derivatives of orders up to $2m$ in (a, b) which are identically zero in neighborhoods of a and of b. Define the inner product $(v, w) = \int_a^b \overline{v(x)} w(x) k(x) dx$. D is said to be semibounded in $\mathfrak{P}$ at the left end point a if there are real numbers c $(a < c < b)$ and λ_0 such that, for all $u(x) \in \mathfrak{P}$ with support contained in (a, c), $(u, Du) \geqq \lambda_0 (u, u)$. Semiboundedness in $\mathfrak{P}$ at the right end point is analogously defined. (If D is semibounded in $\mathfrak{P}$ at either end point, it is known that D then has the decisive property of being semibounded in $\mathfrak{P}$ on the interval $a < x < b$.) Rellich's first criterion states that D is semibounded in $\mathfrak{P}$ at the left end point a if for some real λ the differential equation $Du = \lambda u$ possesses m real solutions such that (1) the zeros in (a, b) of the Wronskian determinant of these m solutions do not accumulate at a, and (2) if $m > 1$, the m solutions are pairwise conjugate. Two solutions y_1 and y_2 are 'conjugate' if $[y_1, y_2] = 0$, the square bracket symbol being appropriately defined from Green's identity $vLu - uLv = (d/dx)[u, v]$, in which $L = k(x)D$. A converse of Rellich's first criterion, apparently due to Heinz, states that, if D is semibounded in $\mathfrak{P}$ at a, then there exist m real, pairwise conjugate solutions of $Du = 0$ whose Wronskian determinant is non-zero for $a < x < b$. Rellich's second semiboundedness criterion is an application of the first criterion to the case of a regular singularity and is formulated in terms of the characteristic polynomial of $L - \lambda k$ at the singular point in question. The starting point in the proofs of these and other results is the problem of factoring L as a product, $L = R^*Q = Q^*R$, of differential operators of mth order, Q^* and R^* being formal adjoints of Q and R, respectively. It is proved that there is such a factoring if and only if: (1) all solutions of $Ru = 0$ and of $Qu = 0$ are also solutions of $Lu = 0$; and (2) the simultaneous relations $Qy = 0$ and $Rz = 0$ imply $[y, z] = 0$. Hence, if there are, in particular, m linearly independent, pairwise conjugate solutions $y_1, \cdots, y_m$ of $Lu = 0$, then $L = Q^*Q$, Q being an mth order differential operator such that $Qy_1 = \cdots = Qy_m = 0$. Rellich's first criterion is a consequence of applying this result to the operator $L - \lambda_0 k$ over a sufficiently narrow interval (a, c) and of integration by parts. This kind of reasoning also leads to an extension of Sturm's oscillation theorem and other results. *A. Douglis* (College Park, Md.)

744:

Feller, William. Differential operators with the positive maximum property. Illinois J. Math. **3** (1959), 182–186.

Let us compare the author's pure second order differential operator A_0 [same J. **1** (1957), 459–504; **2** (1958)

1–18; MR **19**, 1052] with his more general A (perfected in the present paper). (1) $A_0[A]$ is abstracted from the classical aD^2+bD with $a>0$ everywhere [from aD^2+bD+c with $a>0$ and $c\leq 0$ everywhere]. (2) Both A_0 and A have local character. (3) $A_0 f[Af] \leq 0$ whenever f has local max [positive local max]. (4) A_0 [A] is determined by two [three] canonical "parameters": scale x and positive measure m [x and m and another measure g with support perhaps less than the whole basic interval]. (5) A_0 generates an honest diffusion process, i.e. a Markov process with almost all paths continuous [A generates a diffusion process with possible leakage of heat].

The author proves that every strictly positive f in the nullspace of A is convex, and the measure g can be written df'/f.

{An intelligent perturbation of his arguments will supply the infinitesimal generator of a diffusion process that also creates heat, abstracted from aD^2+bD+c without restriction on c, and possessing a still weaker max property.} *H. Mirkil* (Hanover, N.H.)

745:

Doležal, Vladimír. Die asymptotischen Formeln für die Lösungen der Differentialgleichung $y''+f(t)y=0$. Časopis Pěst. Mat. **83** (1958), 451–465. (Czech. Russian and German summaries)

Asymptotic formulas for the general solution $y(t)$ and its derivative $y'(t)$ of the equation $y''+f(t)y=0$ were obtained by M. Zlámal [Czechoslovak Math. J. **81** (1956), 75–91; MR **18**, 210] assuming that on $[t_0, \infty)$, $f \geq \text{const} > 0$ and that $f^{-\frac{1}{2}}$ is convex. The author shows that these formulas remain valid if in these restrictions one replaces $f^{-\frac{1}{2}}$ by $f^{-\alpha}$, $0<\alpha<\frac{1}{2}$. He also constructs an example to show that in general one may not take $\alpha=\frac{1}{2}$.

J. F. Heyda (Cincinnati, Ohio)

746:

Szarski, J.; et Ważewski, T. Sur l'existence des intégrales asymptotiques des équations différentielles issues d'un ensemble de dimension zéro. Colloq. Math. **6** (1958), 215–218.

This paper is concerned with a real differential system

$$(*) \qquad dx/dt = f(x, y, t), \quad dy/dt = g(x, y, t)$$

under the following hypotheses (H): (I) f and g are continuous in an open region W of real (x, y, t)-space, and through each point of W there passes a unique solution of (*); (II) the open tube T: $|x|<1$, $|y|<1$, $-\infty<t<\infty$ and its boundary sets A: $|x|=1$, $|y|\leq 1$, $-\infty<t<\infty$, B_1: $|x|\leq 1$, $y=1$, $-\infty<t<\infty$, B_2: $|x|\leq 1$, $y=-1$, $-\infty<t<\infty$, are such that $T+A+B_1+B_2 \subset W$; (III) $xf(x, y, t)<0$ for $(x, y, t)\in A$, and $yg(x, y, t)>0$ for $(x, y, t)\in B_1+B_2$; (IV) Z is a set such that $Z\subset T+B_1+B_2-A$, $Z(B_1-A)\neq 0$, $Z(B_2-A)\neq 0$; (V) Z is a set of the "type of Antoine", i.e., Z is non-degenerate, compact, totally disconnected, and such that if K is a set homeomorphic to the sphere $x^2+y^2+z^2\leq 1$ with a point of Z interior to K, and also a point of Z exterior to K, then Z has a point in common with the frontier of K. For a system (*) satisfying hypotheses (H) it is shown that Z contains at least one point P: (x_0, y_0, t_0) such that the solution of (*) passing through P is asymptotic with respect to the tube T, that is, the graph of this solution lies in T for $t_0\leq t<\infty$. The presented proof employs an initial argument to show that the set Z may be replaced by a suitable continuum $Z_1 \supset Z$, and the application of earlier results of Ważewski [Ann. Soc. Polon. Math. **20** (1947), 279–313; MR **10**, 122].

W. T. Reid (Evanston, Ill.)

747:

Mikołajska, Z. Remarque sur la structure de l'ensemble engendré par les intégrales asymptotiques d'un système d'équations différentielles. Colloq. Math. **6** (1958), 219–221.

It is shown that the differential system

$$(*) \qquad \frac{dx}{dt} = -xh(x, y), \quad \frac{dy}{dt} = yh(x, y),$$

with $h(x, y)=(x-x_1)^2+(y-y_1)^2$, satisfies with respect to the tube T: $|x|<a$, $|y|<b$, $-\infty<t<\infty$, conditions corresponding to parts I, II, III of hypotheses (H) of the preceding paper by Szarski and Ważewski, while the aggregate of integrals of (*) which are asymptotic with respect to T (for definition see preceding review) form a non-connected set in (x, y, t)-space. The author states that the question of the existence of such a differential system had been raised by T. Ważewski.

W. T. Reid (Evanston, Ill.)

748:

★Kyner, Walter T. Small periodic perturbations of an autonomous system of vector equations. Contributions to the theory of nonlinear oscillations, Vol. IV, pp. 111–124. Annals of Mathematics Studies, no. 41. Princeton University Press, Princeton, N.J., 1958. ix+211 pp. $3.75.

For $i=1, 2, \cdots, k$ let x_i denote an element of an n-dimensional vector space E^n. It is assumed that for each such i the n-dimensional vector differential equation (1) $dx_i/dt = X_i(x_i)$ has a periodic solution with period ω_i. The system to be investigated is the "perturbed" system

$$(2) \qquad \frac{dx_i}{dt} = X_i(x_1, x_2, \cdots, x_k, t, \gamma),$$

where the right member has period T in t and reduces to the right member of (1) as the real parameter γ tends to 0. In the product space $E^{nk}=\prod_{i=1}^{k} E_i^n$ the periodic solutions of (1) form a k-dimensional torus τ. The main object of the paper is to give conditions which are sufficient to guarantee that the solutions of (2) whose initial points ($t=0$) are near enough to τ are periodic. By geometric considerations the problem is reduced to proving the existence of a fixed point for certain transformations which are induced by the transformation connecting the initial values ($t=0$) with the end values ($t=T$) of solutions of (2). It is shown that a fixed point theorem proved by the author in a previous paper [*Contributions to the theory of non-linear oscillations*, Vol. III, pp. 197–205, Princeton Univ. Press, Princeton, N.J., 1956; MR **18**, 408] may be applied provided that (besides differentiability assumptions) the following basic conditions are satisfied: the linear equations of first variation of (1), i.e.,

$$(3) \qquad \frac{dv_i}{dt} = \sum_{j=1}^{n} \frac{\partial X_i(u_i(t))}{\partial x_j} v_j$$

has $n-1$ characteristic exponents with nonzero real parts.

In the case $k=1$ the sufficiency of this condition had been proved by Hufford [Ph.D. Thesis, Princeton, 1953;

Contributions to the theory of nonlinear oscillations, Vol. III, pp. 173–195, Princeton Univ. Press, Princeton, N.J., 1956; MR **18**, 654]. Earlier, Levinson [Ann. of Math. (2) **52** (1950), 727–738; MR **12**, 335] had proved the sufficiency (for $k=1$) under the stronger assumption of $n-1$ characteristic exponents of (3) with non-negative real parts. The case of arbitrary positive integer k had been treated by Marcus [Ph.D. Thesis, University of California, Berkeley, 1954; *Contributions to the theory of nonlinear oscillations*, Vol. III, pp. 17–29, Princeton Univ. Press, Princeton, N.J., 1956; MR **18**, 394]. His result includes that of Levinson but not Hufford's.

E. H. Rothe (Ann Arbor, Mich.)

749:

Yakubovič, V. A. Critical frequencies of quasicanonical systems. Vestnik Leningrad. Univ. **13** (1958), no. 13, 35–63. (Russian. English summary)

Let $M=\{A(t)\}$ denote a given class of complex valued $n\times n$ matrices $A(t)$, $-\infty<t<+\infty$, $A(t+2\pi)=A(t)$, each element of $A(t)$ being L-integrable in $[0, 2\pi]$. If $C\in M$ is a constant matrix, then the frequency $\omega>0$ is said to be critical for the linear system (1) $dx/dt=Cx$, $x=(x_1, \cdots, x_n)$, with respect to the class M if (a) all solutions of (1) (unperturbed system) are bounded in $[0, +\infty)$, (b) for every $\varepsilon>0$ there is some matrix $A(t)\in M$ with $\int_0^{2\pi}\|A(t)-C\|dt<\varepsilon$, so that the (perturbed) linear system (2) $dx/dt=A(\omega t)x$ has unbounded solutions in $[0, +\infty)$.

I. Consider first unperturbed systems of the form $d^2x/dt^2+P_0x=0$, P_0 a real constant symmetric matrix, with real characteristic roots $\omega_j^2>0$, and perturbed systems of the form $d^2x/dt^2+P(\omega t)x=0$, P real and symmetric, i.e., $P_0^*=P_0$, $P^*=P$. The reviewer's quoted result [Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **11** (1941), 633–695; MR **8**, 208] that the only critical frequencies are of the form (3) $|\omega_j\pm\omega_h|/m$, $1\leqq j\leqq h\leqq n$, $m=1, 2, \cdots$, has been improved recently (and its proof simplified) by M. G. Krein, I. M. Gel'fand and V. B. Lidskiĭ [Krein, Collected memoirs dedicated to A. A. Andronov, pp. 413–498, Izdat. Akad. Nauk SSSR, Moscow, 1955; MR **17**, 738; Gel'fand and Lidskiĭ, Uspehi Mat. Nauk (N.S.) **10** (1955), no. 1 (63), 3–40; Amer. Math. Soc. Transl. (2) **8** (1958), 143–181; MR **17**, 482; **19**, 960], who replaced (3) first by $(\omega_j+\omega_h)/m$ for symmetrical systems. In the present paper a new proof of the same result is given. If we waive the condition of symmetry, then the author proves by a very simple argument that all frequencies ω are critical. {For this phenomenon see also the reviewer's paper [loc. cit.], and R. A. Gambill [Riv. Mat. Univ. Parma **6** (1955), 37–43; MR **17**, 849] (unquoted) who gave criteria for instability at any frequency for perturbations of a given form $C+\varepsilon P(\omega t)$, ε a small parameter}.

II. Consider now perturbed systems of the form $d^2x/dt^2+Qdx/dt+P(\omega t)x=0$, $P^*=P$, $Q^*=-Q$, P, Q real (the unperturbed system has $P=P_0$, a constant matrix). According to I. M. Gel'fand and V. B. Lidskiĭ the unperturbed eigenvalues ω_j are divided into eigenvalues of first (ω_j^+) and second (ω_j^-) kind. The author proves that the only critical frequencies are of the form (4) $|\omega_j^+-\omega_h^-|/m$. If for some j, h, $\omega_j^+=\omega_h^-$, then all frequencies ω are again critical.

III. A system $dx/dt=A(t)x$ is said to be quasi-canonical if there is a matrix G with $\det G\neq 0$ such that $(Gx(t), x_1(t))$ = constant for any two solutions $x(t)$, $x_1(t)$. The systems in I and II are quasi-canonical, and so are the systems of the form $\dot p_j=\partial H/\partial q_j$, $\dot q_j=-\partial H/\partial p_j$, $H=\sum a_{jh}(t)p_jp_h+b_{jh}(t)p_jq_h+c_{jh}(t)q_jq_h$. Again the critical frequencies are of the form (4). {A recent independent result of J. K. Hale [Illinois J. Math. **2** (1958), 586–592; MR **20** #3323] along these lines is not quoted. The results of the reviewer, J. K. Hale, R. A. Gambill [see, e.g., Hale, Illinois J. Math. **1** (1957), 98–104; MR **19**, 276] are actually linked by the use of a method which has been applied to nonlinear problems.}

L. Cesari (Baltimore, Md.)

750:

Yakubovič, V. A. On the dynamic stability of elastic systems. Dokl. Akad. Nauk SSSR **121** (1958), 602–605. (Russian)

Some mathematical problems arising in the general theory of the dynamic stability of elastic systems are discussed. [For background on the mathematical problem see #749 above.] Let M be a class of matrices $A(t)\in L(0, 2\pi)$, $A(t+2\pi)=A(t)$. Let C be a constant matrix which is diagonalizable with imaginary diagonal elements. Then all solutions of the unperturbed system (1) $\dot x=Cx$ are bounded as $t\to\infty$. A frequency $\theta>0$ is said to be "resonant" for (1) relative to M if corresponding to each $\varepsilon>0$ there is an $A(t)\in M$, $\int_0^{2\pi}\|A(t)-C\|dt<\varepsilon$, such that the perturbed system $\dot x=A(\theta t)x$ has solutions unbounded as $t\to\infty$. A frequency θ_0 is said to be "critical" for (1) relative to M if for each $\varepsilon>0$, $\delta>0$ there is an $A(t)\in M$, $\int_0^{2\pi}\|A(t)-C\|dt<\varepsilon$, and a number θ, $|\theta-\theta_0|<\delta$, for which (2) has unbounded solutions.

Results are reported giving information on the critical and resonant frequencies and the region of dynamic instability for an unperturbed equation $\ddot y+P_0y=0$ with a perturbed equation $\ddot y+[P_0+\varepsilon Q(\theta t)]y=0$, where P_0 and $Q(\tau)=Q(\tau+2\pi)$ are real symmetric matrices. The criteria given involve expressions that are not difficult to compute.

J. P. LaSalle (Baltimore, Md.)

751:

Opial, Z. Sur une problème de stabilité. Ann. Polon. Math. **5** (1958), 153–164.

Consider (*) $x''+p(t)x=0$ with $p(t)\not\equiv 0$, continuous and periodic of period π. Lyapunov showed that all solutions of (*) are bounded if $p(t)\geqq 0$ and $\int_0^\pi p(t)dt\leqq 4/\pi$, where $4/\pi$ is best possible. The author proves the same contention if $\int_0^\pi p(t)dt\geqq 0$, and either $\int_\tau^{\tau+\pi/4}p^+(t)dt\leqq 2(3-\sqrt{5})/\pi$ for every τ, or $p^+(t)\leqq a^2$ and $\int_\tau^{\tau+2\pi/3}p^+(t)dt\leqq 2/(\frac{1}{3}\pi+a^{-1}\tan\frac{1}{3}\pi a)$ for every τ, where $1<a<3$. He also generalizes in a similar fashion Borg's stability criteria for $p(t)=c^2+q(t)$, where $q(t)$ is continuous and periodic of period π and $0\leqq c\leqq 1$. All proofs are based on certain integral inequalities derived previously by the author [same Ann. **3** (1957), 200–209; MR **19**, 271].

H. A. Antosiewicz (Los Angeles, Calif.)

752:

Bendixson, Ivar. Sur les courbes définies par des équations différentielles. Advancement in Math. **3** (1957), 1–62. (Chinese)

This is a Chinese translation of Bendixson's article of the above title which appeared originally in Acta Math. **24** (1901), 1–88.

Choy-tak Taam (Washington, D.C.)

753:

Peixoto, M. M. On structural stability. Ann. of Math. (2) **69** (1959), 199–222.

Consider the Banach space $\mathfrak{B}$ of all C^1 vector fields

(differential systems) in the n-ball $B^n: \sum_{i=1}^n (X^i)^2 \leq 1$ and let Σ^* be those which point inwards across the boundary S^n. Here use the C^1-metric on vector fields

$$\rho(X, Y) = \max_{B^n} \left(\sum_{i=1}^n |X_i - Y_i| + \sum_{i,j=1}^n \left| \frac{\partial X_i}{\partial x_j} - \frac{\partial Y_i}{\partial x_j} \right| \right).$$

A differential system $X: \dot{x}_i = X_i(x_1, \cdots, x_n)$ in Σ^* is called structurally stable in case there exists $\delta > 0$ such that every system $Y \in \mathfrak{B}$ with $\rho(X, Y) < \delta$ is homeomorphic with X in B^n. In earlier investigations where $n = z$, one defined structural stability by requiring that X and Y were ε-homeomorphic when $\rho(X, Y) < \delta(\varepsilon)$ for prescribed $\varepsilon > 0$. In theorem 1 the author shows that the two definitions of structural stability coincide in the case $n = 2$.

For $n = 2$, the set Σ of all structurally stable differential systems in B^2 is dense in Σ^*.

For $n \geq 2$, the author proves that Σ is an open cone with at most a countable set of components in $\mathfrak{B}$. Also if $X, Y \in \Sigma$ belong to the same component, then there exists a homeomorphism T of B^n onto itself transforming trajectories of X onto trajectories of Y. Conversely, if such a homeomorphism exists, which is furthermore of class C^2 and is C^2-isotopic to the identity, then X and Y belong to the same component of Σ.

L. Markus (Minneapolis, Minn.)

754:

Wang, Rou-hwai. On the existence of bounded solutions of non-linear difference-differential equations. Sci. Record (N.S.) **2** (1958), 23–26.

Consider the real differential-difference system

$$\frac{du(t)}{dt} = A_0 u(t) + A_1 u(t - t_1) + \cdots + A_m u(t - t_m) + f(u(t), u(t - t_1), \cdots, u(t - t_m), t, \varepsilon)$$

for the unknown n-vector $u(t)$; here ε is a small real parameter. Consider the entire function $\det K(\lambda)$ of the complex variable λ, where

$$K(\lambda) = \lambda E - A_0 - e^{-t_1\lambda} A_1 - \cdots - e^{-t_m\lambda} A_m,$$

and assume that it has no root with zero real part.

Then under certain smoothness and growth conditions on f, the author proves that there is a unique, small, bounded solution $u = p(t, \varepsilon)$, on $-\infty < t < \infty$, for each small ε. Also if f is periodic in t, or almost periodic in t, then so is $p(t, \varepsilon)$.

L. Markus (Minneapolis, Minn.)

755:

★Kakutani, S.; and Markus, L. On the non-linear difference-differential equation $y'(t) = [A - By(t-\tau)]y(t)$. Contributions to the theory of nonlinear oscillations, Vol. IV, pp. 1–18. Annals of Mathematics Studies, no. 41. Princeton University Press, Princeton, N.J., 1958. ix+211 pp. $3.75.

The authors give a detailed analysis of the equation figuring in the title, respectively of the equivalent equation

$$(*) \qquad z'(t) = [a - z(t-1)]z(t).$$

These equations, which play some rôle in the theories of growth and of servo-mechanisms, were considered also by other authors such as W. J. Cunningham [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 708–713; MR **16**, 714] and E. M. Wright [J. London Math. Soc. **20** (1945), 68–73; Quart. J. Math. Oxford Ser. (1) **17** (1946), 245–252; J. Reine Angew. Math. **194** (1955), 66–87; MR **7**, 431; **8**, 385; **17**, 272]. As the Addendum of the present authors shows, they were not aware of these latter papers when writing theirs and thus there is some overlap in the results. The main theorems which the authors still consider new, are that every solution of (*) defined for every non-negative t with $z(1) > 0$ and having continuous first derivatives for $t > 1$ will be asymptotic to $z = a$ (i.e., $\lim z(t) = a$, $\lim z'(t) = 0$ hold, and $z(t)$, $z'(t)$ are monotone for large t), if $0 < a \leq 1/e$ and if the interval between the zeros of $z(t) - a$ is at least 1, while no such solutions except $z(t) \equiv a$ will be asymptotic to $z = a$ if $a > 1/e$. Also the cases $a < 0$ and $z(1) < 0$ are considered. The proofs are elementary and can be easily understood.

J. Aczél (Debrecen)

756:

Massera, J. L.; and Schäffer, J. J. Linear differential equations and functional analysis. II. Equations with periodic coefficients. Ann. of Math. (2) **69** (1959), 88–104.

[For part I, see same Ann. **67** (1958), 517–573; MR **20** #3466.] Given a vector-operator equation of the form $x' = A(t)x + f(t)$, where x is an element in a Banach space, a problem of some interest is that of determining properties of the homogeneous equation from those of the solutions of the inhomogeneous equation when $f(t)$ ranges over a suitable set of vector functions. Continuing this investigation begun in their first paper, the authors consider the case in which $A(t)$ is periodic. By means of a generalized Floquet representation which they obtain, they are able to derive a number of results concerning the existence of periodic solutions of the homogeneous equation.

R. Bellman (Santa Monica, Calif.)

757:

Lions, Jacques Louis. Equations différentielles du premier ordre dans un espace de Hilbert. C. R. Acad. Sci. Paris **248** (1959), 1099–1102.

Soit H un espace de Hilbert, $A_1(t)$ une famille ($t \in [0, \mu]$, $\mu < \infty$) d'opérateurs non bornés autoadjoints positivement définis, tels que $A_1^{-1}(t)$ est une fois continûment dérivable au sens faible et vérifie la relation

$$\left| \left(\left(\frac{d}{dt} A_1^{-1}(t) \right) A_1(t)u, A_1(t)u \right) \right| \leq 2\alpha |A_1(t)|^2 + 2c |A_1^{\frac{1}{2}}(t)u|^2$$

où $u \in \mathfrak{D}_{A_1(t)}$ et $\alpha < 1$. Si $A_2(t)$ est une autre famille d'opérateurs non bornés quelconques tels que $\mathfrak{D}_{A_2(t)} \supset \mathfrak{D}_{A_1(t)}$, alors sous des conditions larges et très naturelles sur $A_2(t)$ on obtient la solution de l'équation $u'(t) + A_1(t)u(t) + A_2(t)u(t) = f(t)$, où $u'(t)$ et $f(t)$ (arbitraire) sont des fonctions de carré sommable sur $[0, \mu]$ à valeurs dans H. L'unicité est assurée sous des conditions un peu plus fortes sur $A_2(t)$. On donne aussi une construction qui conduit aux familles fondamentales $A_1(t)$. Ce sont quelques-uns des premiers résultats dans cette étude qui n'exigent pas la constance des domaines des opérateurs non bornés et par leur forme ils donneront beaucoup de suggestions au lecteur.

C. Foiaş (Bucharest)

758:

Huet, Denise. Phénomènes de perturbation singulière. C. R. Acad. Sci. Paris **247** (1958), 2273–2276.

Comportement des solutions de l'équation $(\varepsilon A(t) + B(t) + d/dt)u_\varepsilon = f$ [resp. $+ d^2/dt^2$], $u_\varepsilon(0) = 0$ [et $u_\varepsilon'(0) = 0$ s'il y a un terme en d^2/dt^2], lorsque $\varepsilon \to 0$; $A(t)$ et $B(t)$ sont des opérateurs non bornés dans un espace de Hilbert, $A(t)$ étant "plus fort" que $B(t)$.

J. L. Lions (Nancy)

PARTIAL DIFFERENTIAL EQUATIONS

See also 892, 1085, 1101, 1138.

759:

Turpin, Maurice. Équation de Monge-Ampère à trois variables. C. R. Acad. Sci. Paris **248** (1958), 60–62.

In the space $(x_1, x_2, x_3, p_1, p_2, p_3, z)$, set $\omega = dz - p_1dx_1 - p_2dx_2 - p_3dx_3$. On a domain in the above space, let $\theta_1, \theta_2, \theta_3$ be distinct analytic pfaffian forms in $dx_1, dx_2, dx_3, dp_1, dp_2, dp_3$. A system of partial differential equations for the unknown function z in independent variables x_1, x_2, x_3 is called equations of Monge-Ampère, when it is equivalent to an exterior differential system of the form: $\omega = 0$, $\theta_1 \wedge \theta_2 \wedge \theta_3 = 0$. The writer announces theorems on such systems concerning the existence of solutions (reduced to the theory of E. Cartan on systems in involution), the characteristic systems, and in a certain special case, methods of integration. *M. Kuranishi* (Nagoya)

760:

Osborn, Howard. The existence of conservation laws. I. Ann. of Math. (2) **69** (1959), 105–118.

Let $A = (a_j^i)$ be a matrix of degree n. For a pfaffian form $\theta = \theta_i dx^i$ on a domain $\mathcal{N}$ containing the origin in a real n-dimensional vector space $E' = \{(x^1, \cdots, x^n)\}$, set $A(\theta) = \theta_i a_j^i dx^j$. The problem, motivated by the theory of partial differential equations, is to determine locally θ such that both θ and $A(\theta)$ are exact. Such θ is called a conservative law for A. We identify the tangent vector spaces at points (x) to E, the dual of E'. Consider A as an endomorphism of E'. Denote by $\mathfrak{A}$ the ring of endomorphisms generated by A and scalar multiplications. A is called cyclic when there is w in E' such that $E' = \mathfrak{A}w$. For a subspace F of E', set $\mathfrak{I}(F) = \{B \in \mathfrak{A};\ BF = 0\}$. Then by the classical decomposition theorem, $E' = E_1' \oplus \cdots \oplus E_q'$, where $AE_i' \subset E_i'$, the restriction of A to E_i' is cyclic, $\dim E_i' \leq \dim E_j'$ and $\mathfrak{I}(E_i') \supseteq \mathfrak{I}(E_j')$ for $i \leq j$. Take $w^{(i)}$ in E_i' such that $w^{(i)}$ generates E_i' with respect to A. Set $\mathscr{W}_h$ the intersection of $\mathcal{N}$ and the subspace generated by $w^{(h)}, \cdots, w^{(q)}$. The writer proves the following: (I) Let $\delta(A)$ be the degree of the minimum polynomial of A. Effectively corresponding to germs of any $\delta(A)$ real analytic functions of q variables, there exists at least one conservative law for A. (II) Let $E = E_1 \oplus \cdots \oplus E_q$ be the decomposition of E dual to the above decomposition of E'. An analytic conservative law, considered as a function on $\mathcal{N} \times E$, is uniquely determined by its restriction to $\mathscr{W}_1 \times E_1, \cdots, \mathscr{W}_q \times E_q$. The proof for (I) is to construct an involutive exterior differential system such that its integrals represent conservative laws and apply the Cartan-Kahler theorem. Algebraic theorems concerning symmetric p-multilinear functions $f(u_1, \cdots, u_p)$ such that $f(u_1, \cdots, u_{p-1}, Au_p)$ are symmetric play a fundamental role in the proof of the above theorems.

M. Kuranishi (Nagoya)

761:

Volpato, Mario. Sul problema di Cauchy per una equazione lineare alle derivate parziali del primo ordine. Rend. Sem. Mat. Univ. Padova **28** (1958), 153–187.

Consider

$$\text{(I)} \qquad z_x + \sum_{s=1}^{n} f_r(x, y, \xi, \lambda) z_{y_r} = f_{n+1}(x, y, \xi, \lambda),$$

$y = (y_1, \cdots, y_n)$; assume the f_r $(r = 1, \cdots, n+1)$ defined in $a \leq x, \xi \leq b$, $(y, \lambda) \in R^{n+1}$, summable in x for fixed (y, ξ, λ) and such that for almost all x they satisfy

$$|f_r(x, y, \xi, \lambda) - f_r(x, y', \xi', \lambda')| \leq P_r(x)\left(\|y - y'\| + \left|\int_\xi^{\xi'} \mu(\tau)d\tau\right| + |\lambda - \lambda'|\right),$$

P_r, μ being non-negative summable functions. Let $\phi(y, \lambda)$ be a Lipschitzian function defined in R^{n+1}. Then there exists a unique continuous function $z(x, y, \xi, \lambda)$ with $z(\xi, y, \xi, \lambda) = \phi(y, \lambda)$, such that it satisfies a Lipschitz condition with respect to (y, λ), is absolutely continuous with respect to (x, ξ), the partials z_x, z_ξ being summable, and satisfies (I). Other regularity properties of z are also proved, some of them under supplementary assumptions on the f_r, ϕ. *J. L. Massera* (Montevideo)

762:

Fox, David W. The solution and Huygens' principle for a singular Cauchy problem. J. Math. Mech. **8** (1959), 197–219.

The subject of this paper is a study of a singular Cauchy problem for the hyperbolic equation

$$\text{(1)} \qquad \frac{\partial^2 u}{\partial t^2} + \frac{k}{t}\frac{\partial u}{\partial t} - \sum_{i=1}^{m}\left(\frac{\partial^2 u}{\partial x_i^2} + \frac{\lambda_i}{x_i^2}u\right) = 0,$$

where k and λ_i are parameters, usually real, and where the exceptional values $k = -1, -3, -5, \cdots$ of Weinstein are excluded. The data are $u = f(x_1, \cdots, x_m)$, $u_t = 0$ when $t = 0$.

Following a method of Bureau, the author constructs a solution

$$\text{(2)} \qquad \frac{\Gamma(\frac{1}{2}k + \frac{1}{2})}{\pi^{\frac{1}{2}m}\Gamma(\frac{1}{2}k + \frac{1}{2} - \frac{1}{2}m)} \int_S f(\xi_1, \cdots, \xi_m) v d\xi_1, \cdots, d\xi_m,$$

where S is the closed m-dimensional sphere of radius $|t|$ and centre $(x_1, x_2, \cdots, x_m)$ in the initial hyperplane, and v is a particular solution of (1) of the form $|t|^{1-k}\Gamma^{\frac{1}{2}(k-m-1)} \times F(z_1, z_2, \cdots, z_m)$, with $\Gamma = t^2 - \sum_{i=1}^{m}(x_i - \xi_i)^2$, $z_i = \Gamma/(4x_i\xi_i)$ and F one of Lauricella's generalized hypergeometric functions whose coefficients depend on the parameters k and λ_i.

It is proved that (2) is a solution of (1) for Re $k \geq m + 3$ when f is a continuous function, and that it satisfies the initial conditions when f is continuously differentiable. The solution is continued analytically into Re $k \geq m - 2n + 1$ where n is a non-negative integer on the assumption that f has continuous derivatives of order $n + 2$, the exceptional values being excluded.

Uniqueness, for $k > 0$, is discussed by means of the techniques of Zaremba and of Friedrichs and Lewy, the results being similar to those obtained by W. Walter [Math. Z. **67** (1957), 361–376; MR **19**, 1059] for the Euler-Poisson-Darboux equation to which (1) reduces when all the λ_i are zero.

Lastly, a necessary and sufficient condition for Huygens' principle is obtained in the form $k = m - 2n - 1$ $(n \in \{0, 1, 2, \cdots\})$, $\lambda_i = \alpha_i(1 - \alpha_i)$ $(\alpha_i \in \{1, 2, \cdots, n+1\}$, $\sum_{i=1}^{m} \alpha_i \leq m + n)$. *E. T. Copson* (St. Andrews)

763:

Chow, Tse-Sun. On the solution of certain differential equations by characteristic function expansions. Quart. Appl. Math. **16** (1958), 227–235.

This article is concerned with the solution of the parabolic equation

$$p_0\frac{\partial^2 u}{\partial x^2}+p_1\frac{\partial u}{\partial x}+p_2 u-\frac{\partial u}{\partial t}=h(x,t)$$

in $a\leqq x\leqq b$, $t\geqq 0$, where $h(x,t)$ is identically zero, p_0, p_1, p_2 are functions of x alone, where u satisfies the initial condition $u(x,0)=u_0(x)$ and the non-homogeneous boundary conditions

$$\alpha_1 u(a,t)+\alpha_2 u(b,t)+\alpha_3 u_x(a,t)+\alpha_4 u_x(b,t)=f(t),$$
$$\beta_1 u(a,t)+\beta_2 u(b,t)+\beta_3 u_x(a,t)+\beta_4 u_x(b,t)=g(t),$$

the α_i and β_i being constants. By a change of the dependent variable, the boundary conditions are made homogeneous ($f\equiv 0$, $g\equiv 0$) at the price of making h different from zero.

The solution is then obtained in the form

$$\sum_1^\infty F_n(t)\psi_n(x)$$

where the functions $\psi_n(x)$ are the eigenfunctions of the operator

$$-p_0\frac{d^2}{dx^2}-p_1\frac{d}{dx}-p_2$$

under the boundary conditions

$$\alpha_1\psi(a)+\alpha_2\psi(b)+\alpha_3\psi'(a)+\alpha_4\psi'(b)=0,$$
$$\beta_1\psi(a)+\beta_2\psi(b)+\beta_3\psi'(a)+\beta_4\psi'(b)=0.$$

As an example, the axially-symmetric diffusion equation

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{2}\frac{\partial u}{\partial r}=\frac{\partial u}{\partial t}$$

is discussed. The analysis is formal.

E. T. Copson (St. Andrews)

764:

Grigor'eva, E. A. The straight line method in mixed problems for parabolic systems. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 648–651. (Russian)

For the system of equations

$$(1)\qquad \frac{\partial u_i}{\partial t}=a_i\frac{\partial^2 u_i}{\partial x^2}\quad (i=1,\cdots,m),\quad a_i=\text{const.},\ a_i>0,$$

the author shows that the solution of the initial and boundary value problem

$$(2)\qquad u_i(x,0)=0,$$

$$(3)\qquad \begin{cases}\sum_{j=1}^m a_{1,j}u_j(0,t)+\dfrac{\partial u_1(0,t)}{\partial x}=f_1(t),\\ \sum_{j=1}^m a_{m+1,j}u_j(l,t)+\dfrac{\partial u_1(l,t)}{\partial x}=f_{m+1}(t),\quad a_{k,j}=\text{const.},\end{cases}$$

can be obtained as the uniform limit, as $h\to 0$, of solutions of the system

$$\frac{1}{h}[u_{1h}(x,t+h)-u_{1h}(x,t)]=a_i\frac{\partial^2 u_{1h}(x,t+h)}{\partial x^2}$$

with corresponding initial and boundary conditions. The proof is based on the known solution of (1), (2), (3) by integral equations, and on a theorem of Sobolev [Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 413–436; MR **18**, 322] on the inversion of operators which suitably approximate a given completely continuous operator. The exposition is clear and concise. *R. Finn* (Stanford, Calif.)

765:

Friedman, Avner. Convergence of solutions of parabolic equations to a steady state. J. Math. Mech. **8** (1959), 57–76.

This paper is divided into three parts. Part I is concerned with the equation $L(u)=f(x,t)$, where L is a linear parabolic differential operator of the second order with coefficients depending on x and t. Here x is a point in n-dimensional Euclidean space. The underlying domain is a cylinder with base on $t=0$, contained in $t>0$. The following results are proved: (a) If $f\to 0$ as $t\to\infty$, and if $u\to 0$ as $t\to\infty$ on the lateral boundary of the cylinder, then $u\to 0$ inside the cylinder; (b) if the coefficients of L tend to limits as $t\to\infty$, and if u tends to a limit as $t\to\infty$ on the lateral boundary of the cylinder, then u tends to a limit v inside the cylinder as $t\to\infty$. The limiting function v satisfies the equation $(L'+\partial/\partial t)v=f'$, where L' and f' are the limits of L and f as $t\to\infty$.

In part II the results of part I are generalized to the non-linear equation $L(u)=f(x,t)+g(x,t,u)$. Part III is devoted to the extension of the previous results to domains whose cross section at $t=\tau$ becomes unbounded as $\tau\to\infty$. *J. Elliott* (New York, N.Y.)

766:

Friedman, Avner. Asymptotic behavior of solutions of parabolic equations. J. Math. Mech. **8** (1959), 387–392.

Let $u(x,t)$ be a solution of a parabolic, second order, linear partial differential equation $L(u)=f(x,t)$, in a cylinder contained in $t>0$ with a bounded n-dimensional base B, under the boundary condition $u(x,t)=h(x,t)$ for $x\in$ bdry B, $t>0$. This paper is a continuation of the paper by the author reviewed above, the results of which are employed to investigate the asymptotic behavior of $u(x,t)$ as $t\to\infty$, given the asymptotic behavior of f, h, and the coefficients of L. *J. Elliott* (New York, N.Y.)

767:

Cattabriga, Lamberto. Un confronto fra operatori generalizzanti l'operatore $\partial^2/\partial x^2+\partial/\partial y$. Ricerche Mat. **7** (1958), 64–70.

Three weak versions of the parabolic operator $\partial^2/\partial x^2+\partial/\partial y$, one of which has been studied by P. Hartman and A. Wintner [Amer. J. Math. **75** (1953), 598–610; MR **15**, 227] and two of which have been devised by B. Pini [Boll. Un. Mat. Ital. **9** (1954), 244–250; MR **16**, 698], are investigated. Conditions under which all three versions are identical are proved. The principal tool is a theorem on abstract sub-additive operators due to W. Kozakievicz [Rocz. Towarz. Nauk. Warszaw. **26** (1933)].

C. R. DePrima (Pasadena, Calif.)

768:

Haimovici, Adolf. Une généralisation de la méthode de Fourier de résolution de certains problèmes aux limites. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) **2** (1956), 133–143. (Romanian. Russian and French summaries)

Consider the equation

$$(10)\qquad u_{xx}+cu_{tt}+a(x)u_x+b(t)u_t=0$$

with the non-linear boundary conditions

$$u_x(0,t)=\alpha(t)V[u(0,t)],\quad u_x(1,t)=\beta(t)V[u(1,t)],$$

where V is a continuous function with a bounded derivative such that $\int dz/V(z)$ exists in the appropriate interval. The problem is reduced in the general case to an equivalent one which is not reproduced here for the sake of brevity. If $a=c=0$, $b=-1$ (heat equation), the existence of a solution is established provided the bounds of α, β, V', etc., satisfy a certain inequality; a supplementary condition ensures uniqueness. (From the author's summary)

J. L. Massera (Montevideo)

769:

Devingtal', Yu. V. Application of the method of successive approximations to a type of singular integral equations in connection with the solution of the generalized Tricomi problem for the equation of M. A. Lavrent'ev. Uspehi Mat. Nauk **14** (1959), no. 1 (85), 169–176. (Russian)

Let D be the domain in the (x, y) plane which is bounded by the upper half circle σ: $(x-\frac{1}{2})^2+y^2=\frac{1}{4}$, $y\geq 0$, and in the lower half plane by $y=x-1$ and a smooth curve $y=-\gamma(x)$ $(0\leq x\leq l<1)$ with $\gamma(0)=0$, $-\gamma(l)=l-1$ and $-1\leq\gamma'(x)<1$. The "generalized Tricomi problem" consists in finding a function $u(x, y)$ with certain regularity properties in $\bar{D}$ which vanishes on σ, equals $\Phi_1(x)$ on $y=-\gamma(x)$ where $\Phi_1(x)$ is a given function with $\Phi_1(0)=0$ satisfying a Hoelder condition, and which in the interior points of D (except those for which $y=0$) is a solution of the equation

$$u_{xx}+\operatorname{sign} y\,u_{yy}=0. \tag{1}$$

This problem had been treated by A. B. Bicadze [*K probleme uravnenii smešannogo tipa*, Trudy Mat. Inst. Steklov, vol. 41, Izdat Akad. Nauk SSSR, Moscow, 1953; MR **16**, 43].

In the present paper the author shows that it is possible to reduce this problem to the solution of the "special" Tricomi problem for which $\gamma(x)=x$. In order to do this he considers the problem of expressing in terms of $\gamma(x)$ and $\Phi_1(x)$ the values $\varphi_1(x)$ which a solution $u(x, y)$ of the generalized problem takes on $y=-x$. It is shown that this latter problem leads to an integral equation of the form

$$\varphi(x)+\int_0^1 \frac{K(\lambda(x), t)}{t-\lambda(x)}\,\varphi(t)dt=\Phi(x), \tag{2}$$

where λ and Φ are determined in terms of γ and Φ_1, and where $K(\lambda, t)$ is given explicitly. Under the assumptions made for the original problem the author shows by a modification of the usual procedure of successive approximation that (2) admits one and only one solution provided that the curve $y=-\gamma(x)$ differs from the characteristic $y=-x$ of (1) by not too much in a certain x-interval $0\leq x\leq q/2$.

E. H. Rothe (Ann Arbor, Mich.)

770:

Feller, William. On the equation of the vibrating string. J. Math. Mech. **8** (1959), 339–348.

As Feller's abstract diffusion operator Ω [Illinois J. Math. **1** (1957), 459–504; MR **19**, 1052] generalizes d^2/dx^2, so $\Omega u=d^2u/dt^2$ generalizes the classical one-dimensional wave equation. Feller here argues that this generalized version arises naturally from simple physical assumptions (in contrast with imprecise, even self-contradictory, classical derivation). He interprets the canonical "parameters" and boundary conditions of Ω in language appropriate to the vibrating string. (Only some of the possible boundary conditions have been recognized classically.) In the final section, Feller sets up a Hilbert space E (the square of the E norm is energy) within which he describes by means of a one-parameter unitary operator group the dependence of solutions on initial conditions. Like other hyperbolic equations, Feller's equation fails to have any regularizing effect on a solution as t increases. But it has the compensating advantage that "arbitrary" initial conditions (both positions and velocities) occur, and these determine not only the future but also the past.

H. Mirkil (Hanover, N.H.)

771:

Eidel'man, S. D. The behaviour of the solutions of a parabolic system in the neighbourhood of an isolated singular point. Dokl. Akad. Nauk SSSR **125** (1959), 743–745. (Russian)

The author studies the isolated singularities of solutions of general systems of parabolic equations satisfying some smoothness assumptions. He proves that any such solution satisfying certain boundedness conditions is necessarily a finite linear combination of a regular solution and space-derivatives of a given fundamental matrix. He uses this result to establish uniqueness of the essential part of the fundamental matrix. The results of this paper form a generalization of the special case of the heat equation considered by the author in Uspehi Mat. Nauk (N.S.) **11** (1956), no. 3 (69), 207–210 [MR **19**, 557]; the methods are the same.

A. Friedman (Berkeley, Calif.)

772:

Aržanyh, I. S. Integral representation of solutions of a system of differential equations of field theory. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **21** (1957), 7–28. (Russian)

The author is concerned with systems of partial differential equations, in three space dimensions and one time, of the form

$$\operatorname{div} v_k=\theta_k,$$

$$\operatorname{rot} v_k=\omega_k+\sum_{l=1}^{N} a_{kl}\frac{\partial v_l}{\partial t}\quad (k=1, 2, \cdots, N),$$

where θ_k and ω_k are given functions of the coordinates x_1, x_2, x_3, in a domain Q with boundary S, and of the time t; and the coefficients a_{kl} are constants. The main purpose is to obtain integral representations (i.e., in terms of integrals extended over Q and S) for solutions of systems of this type. In matrix form, such a system may be written $\operatorname{div} V=\Theta$, $\operatorname{rot} V=\Omega+A\partial V/\partial t$, and can be put into one of four canonical forms, depending on the eigenvalues of the N by N square matrix A. Applying previous results [same Trudy **13** (1954), 19–41; **16** (1955), 5–22; Dokl. Akad. Nauk SSSR **100** (1955), 1053–1056; MR **18**, 808; **20** #4951; **16**, 1181] the author obtains some integral representations of the desired form. These are, in turn, employed to study the boundary value problems where the normal components of the first P (vector valued) functions are assigned on the boundary S: $n\cdot v_p(s; t)=f_p(s; t)$ $(p=1, \cdots, P)$; and the tangential components of the remaining functions are assigned on the boundary S: $n\times v_q(s; t)=g_q(s; t)$ $(q=P+1, P+2, \cdots, N)$; where s denotes any point of S.

J. B. Diaz (College Park, Md.)

773:

Peetre, Jaak. Une classe d'opérateurs différentiels. C. R. Acad. Sci. Paris **248** (1959), 1102–1103.

Une extension des résultats de Hörmander [Acta Math. **94** (1955), 161–248; MR **17**, 853]. Soit $P(x, D)$ un opérateur différentiel à coefficients variables. Supposons que l'ensemble des opérateurs tangentiels $P(x_0, D)$ attachés à chaque point x_0 forme des opérateurs différentiels (à coefficients constants) également forts dans la terminologie de Hörmander. Pour tel opérateur l'auteur énonce quelques résultats. Un d'eux est le suivant: Tout point x admet un voisinage ouvert $\mathcal{O}_x$ tel que, si $f \in \mathscr{D}'$ a son support contenu dans $\mathcal{O}_x$, alors $P(x, D)f \in \mathscr{H}^s$ implique $Mf \in \mathscr{H}^s$, où M est un opérateur également fort qu'un des $P(x_0, D)$.

S. Mizohata (Kyoto)

774:

Fridlender, V. R. Analytic solutions of the Cauchy problem for certain non-linear partial differential equations. Mat. Sb. (N.S.) **47** (89) (1959), 17–44. (Russian)

The present paper is devoted to an extension to more general equations of the considerations of G. S. Salehov [Doctoral dissertation, Kazan, 1946] and the author [Doctoral dissertation, Kazan, 1953]. For a detailed bibliography on the subject, see G. S. Salehov and V. R. Fridlender, Uspehi Mat. Nauk (N.S.) **7** (1952), no. 5 (51), 169–192 [MR **15**, 227]. The principal aim is to strengthen S. Kowalevski's classical results concerning the equations studied. Consider the classical theorem of S. Kowalevski, which asserts the existence of a solution, analytic in (t, x), of the Cauchy problem for the partial differential equation

$$(*) \qquad \frac{\partial^p u}{\partial t^p} = F\left(t, x, u, \cdots, \frac{\partial^{r+q} u}{\partial t^r \partial x^q}\right),$$

provided that: (1) the equation is "normal" in the sense of Kowalevski, that is, $(p-r)/q \geqq 1$, $p > r$; (2) the initial Cauchy data φ_k:

$$\left.\frac{\partial^k u}{\partial t^k}\right|_{t=0} = \varphi_k(x) \quad (k = 0, 1, \cdots, p-1),$$

are analytic in x in a neighborhood of $x = x_0$; and (3) the function F is analytic in all its arguments in the neighborhood of the point: $t = 0$, $x = x_0$, $u = \varphi_0(x_0)$, $\cdots$, $\partial^{r+q} u/\partial t^r \partial x^q = \varphi_r^{(q)}(x_0)$. Definition: A member S is a "sufficient weight" of (*) provided it is true that if the initial Cauchy data φ_k belong to Gevrey's class δ, then the Cauchy problem for (*) with these initial data possesses a solution which is analytic in t at $t = 0$. Theorem 1: If $\delta \geqq 1$, where

$$\delta \equiv \frac{p - [r - (s-1)\min r] - \varepsilon(s-1)}{q - (s-1)\min q}$$

and $\varepsilon > 0$ is arbitrary, then δ is a sufficient weight for the equation

$$\frac{\partial^p u}{\partial t^p} = \frac{\partial^{r_1+q_1} u}{\partial t^{r_1} \partial x^{q_1}} \cdots \frac{\partial^{r_s+q_s} u}{\partial t^{r_s} \partial x^{q_s}}.$$

Here the following notation has been used:

$$r = \sum_{i=1}^{s} r_i; \quad q = \sum_{i=1}^{s} q_i; \quad \min r = \min_{i=1,\cdots,s} r_i;$$

$$\min q = \min_{i=1,\cdots,s} q_i; \quad \max r = \max_{i=1,\cdots,s} r_i < p.$$

The author also proves a theorem of the same nature for the Cauchy problem for the equation

$$\frac{\partial^p u}{\partial t^p} = \sum_{k=1}^{N} f_k(t, x) \frac{\partial^{r_{1k}+q_{1k}} u}{\partial t^{r_{1k}} \partial x^{q_{1k}}} \cdots \frac{\partial^{r_{sk}+q_{sk}} u}{\partial t^{r_{sk}} \partial x^{q_{sk}}} + f_0(t, x),$$

and for a certain system of equations. The Goursat problem for the equation

$$\frac{\partial^p u}{\partial t_1^{p_1} \cdots \partial t_k^{p_k}} = f(t_1, \cdots, t_k; x_1, \cdots, x_n) \prod_{i=1}^{s} \frac{\partial^{r_i+q_{in}} u}{\partial t_1^{r_{i1}} \cdots \partial t_k^{r_{ik}} \partial x_1^{q_{i1}} \cdots \partial x_n^{q_{in}}}$$

is also treated from the same point of view.

J. B. Diaz (College Park, Md.)

POTENTIAL THEORY

See also 718, 722.

775:

Balakrishnan, A. V. Representation of abstract Riesz potentials of the elliptic type. Bull. Amer. Math. Soc. **64** (1958), 288–289.

A representation of the Riesz potentials of elliptic type is obtained using semi-group theory. Let $\xi = (\xi_1, \cdots, \xi_n)$, $-\infty < \xi_k < \infty$, and let

$$(1) \qquad T(\xi) = \prod_{k=1}^{n} T_k(\xi_k),$$

where each $T_k(\xi_k)$ is a strongly continuous one-parameter semi-group over a Banach space X with infinitesimal generator A_k. Let (2) $C = \sum_{k=1}^{n} A_k^2$. The Riesz potential $(-C)^\alpha$, $0 < \operatorname{Re} \alpha < 1$, is shown to have the representation

$$(3) \quad (-C)^\alpha = \frac{4^\alpha \Gamma(\alpha + n/2)}{\pi^{n/2} \Gamma(-\alpha)} \int_{E_n^+} [T(\xi)x + T(-\xi)x - 2x] |\xi|^{-2\alpha-n} dV_\xi,$$

where E_n^+ is the region in which $\xi_k \geqq 0$ for all k, $x \in$ domain C, and dV_ξ is the n-dimensional volume element.

J. Elliott (New York, N.Y.)

776:

Ninomiya, Nobuyuki. Étude sur la théorie du potentiel pris par rapport au noyau symétrique. J. Inst. Polytech. Osaka City Univ. Ser. A **8** (1957), 147–179.

L'A. considère dans un espace localement compact Ω un noyau $K(x, y)$ symétrique $\geqq 0$ continu, fini sauf peut-être pour $x = y$. Il étudie essentiellement les relations entre les principes d'energie, d'équilibre, de balayage et les principes de maximum. Comme la propriété d'énergie $\geqq 0$ équivaut à $(\mu, \nu)^2 \leqq I(\mu) I(\nu)$ (énergie mutuelle $(\mu, \nu) = \int U^\nu d\nu$, énergie $I(\mu) = \int U^\nu d\mu$) pour tout couple de μ, $\nu \geqq 0$ et seulement lorsque μ et ν chargent des boréliens disjoints, l'A. introduit $G(\mu, \nu) = I(\mu) I(\nu)/(\mu, \nu)^2$ et utilise des couples pouvant le rendre minimum parmi des μ, ν chargeant respectivement des ensembles disjoints donnés. Ce nouvel instrument lui donne les résultats suivants.

(a) Pour un noyau, la propriété que l'énergie est $\geqq 0$ [respectivement, qu'elle soit en plus nulle seulement si la mesure est 0] équivaut à ce que, pour des μ, ν d'énergie finie à support compact, la condition $U^\mu \leqq U^\nu$ sur le support S_μ entraîne l'inégalité en un point de S_ν [resp. entraîne, si

$\mu \neq \nu$, l'inégalité stricte sur un ensemble de ν-mesure >0]. De là résulte que l'énergie est ≥ 0 (c.à.d. le noyau est de type positif) en particulier si K satisfait au premier principe du maximum (Frostman) ou au 2e (de Cartan, dit aussi principe de domination d'un potentiel d'énergie finie par un potentiel quelconque).

(b) L'A. considère ensuite les principes d'équilibre et du balayage pour un compact, sous la forme ordinaire, avec la condition d' "à peu près partout" (p.p.p.) dans le sens de "sauf sur un ensemble de mesure nulle pour toute mesure d'énergie finie". Il montre que le principe d'équilibre et du balayage équivalent respectivement au 1er et 2ème principe du maximum, les potentiels-solutions coïncidant 2 à 2 p.p.p.

(c) Les deux principes du maximum sont équivalents aux conditions respectives que pour toute mesure $\lambda \geq 0$, avec S_λ compact $\not\ni \xi$, la condition $U^\lambda \leq K(x, \xi)$ sur S_λ entraîne $\lambda(\Omega) \leq 1$ ou respectivement la même condition partout. Étude plus approfondie dans R^n.

(d) L'A. recherche enfin des critères pour l'unicité de la mesure de potentiel donné, c.à.d. pour que $U^\mu = U^\nu$ p.p.p. entraîne $\mu = \nu$ (μ et ν quelconques mais à supports compacts).

Enfin l'A. adapte les résultats précédents à un noyau de signe quelconque, en vue d'application au noyau logarithmique. *M. Brelot* (Paris)

777:

Endl, Kurt. Zum Dirichletschen Problem auf den Greenschen Linien. Ann. Acad. Sci. Fenn. Ser. A. I. no. **251/3** (1958), 14 pp.

L'A. développe une partie de la théorie résumée dans une note [C. R. Acad. Sci. Paris **244** (1957), 1705–1707; MR **19**, 406] et nous renvoyons à l'analyse. Il s'agit d'un problème de Dirichlet dans un espace de Green, où la condition aux limites consiste en une certaine convergence en moyenne vers une fonction (dite radiale) définie sur l'ensemble L des lignes de Green (issues d'un pôle), pourvu d'une topologie "naturelle" et d'une "mesure de Green" dg. Il y a au plus une fonction harmonique "indifférente" (solution) admettant comme radiale (dite fondamentale) une fonction φ donnée sommable-dg; les φ ayant une telle solution sont les fonctions sommables pour une autre mesure dg^*. On étudie sur L les ouverts dont la fonction caractéristique admet une solution; ils définissent une topologie. Les fonctions sommables-dg^*, semi-continues dans la topologie naturelle le sont dans la nouvelle topologie. Les fonctions (sommables-dg) semi-continues dans la nouvelle topologie le sont dans l'ancienne.

M. Brelot (Paris)

778:

Choquet, Gustave. Capacitabilité en potentiel logarithmique. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 321–326.

L'A. étudie dans le plan la notion de capacité, relative au noyau logarithmique. Considérons comme "négligeable" tout ensemble dont tout sous-ensemble compact ne peut porter une charge >0 de potentiel logarithmique borné supérieurement. Prenons un compact K non négligeable; on sait qu'il y existe une mesure ≥ 0 unique ν ($\nu(K)$ étant supposé égal à 1), dont le potentiel vaut sur K une constante h, sauf sur un ensemble négligeable. On appelle ordinairement capacité logarithmique de K le nombre e^{-h} {et non $1/h$ comme le dit l'A.}.

L'A. montre que $(-h)$ considéré comme fonction de K est, selon les définitions de sa théorie générale de la capacité, une capacité alternée d'ordre infini. Il utilise pour cela un disque à rayon croissant et la capacité relative à la fonction de Green correspondante. De là résulte que dans le plan, tout ensemble borélien ou analytique est "capacitable" pour les diverses capacités, $(-h)$, $1/h$ ou e^{-h}. Ainsi est étendue à la capacité logarithmique (ou diamètre transfini) la propriété fondamentale de capacitabilité.

M. Brelot (Paris)

779:

Erohin, V. The connection between metric dimension and harmonic capacity. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 81–88. (Russian)

The author takes the "n-dimensional capacity" $c_n(E)$ of a Borel set E, in Euclidean n-dimensional space, to be the supremum of the total mass in a non-negative mass-distribution over E for which the (appropriate) potential nowhere exceeds 1. (In particular, the potential is Newtonian [logarithmic] when $n=3$ [2].) Also, denoting by $\phi(\rho)$ a function that is non-negative, (broadly) increasing, and continuous on $(0, 1)$, and by S_k a sphere with radius ρ_k, he considers the generalized Hausdorff measure $m_\phi(E)$ that equals $\liminf_{\varepsilon \to 0} \sum_{k=1}^{\infty} \phi(2\rho_k)$ [$E \subset \bigcup_{k=1}^{\infty} S_k$, $\rho_k \leq \varepsilon$ $(k = 1, 2, \cdots)$]. He proves the following theorems. 1. Let

$$(*) \qquad \int_0^1 \rho^{1-n} \phi(\rho) d\rho < \infty.$$

Then $c_n(E) > 0$ if $m_\phi(E) > 0$. 2. As $\rho \to 0$ let

$$(**) \qquad \liminf \phi(\rho) \rho^{2-n} > 0 \quad (n \geq 3),$$
$$\liminf \phi(\rho) \log 1/\rho > 0 \quad (n = 2).$$

Then $c_n(E) = 0$ if $E = E_1 \cup E_2 \cup \cdots$ and $m_\phi(E_k) < \infty$ $(k = 1, 2, \cdots)$. It follows, in particular, that if the (Hausdorff) metrical dimensionality $r(E)$ of E exceeds $n-2$ then $c_n(E) > 0$, but if $r(E) < n-2$ then $c_n(E) = 0$. The author indicates that theorem 1 [2] would be false if the condition (*) [(**)] were omitted. {Remark: In the case where $n=2$ and E is closed the author's results were known [cf. R. Nevanlinna, *Eindeutige analytische Funktionen*, 2nd ed., Springer, Berlin–Göttingen–Heidelberg, 1953; MR **15**, 208; pp. 155–163]; in the case where $n \geq 2$ and $\phi'(\rho)$ is continuous and positive, his theorem 1 is implicit in a lemma due to H. Cartan [Ann. Sci. École Norm. Sup. (3) **45** (1928), 256–346; especially §§ 16–18, pp. 277–282].}

H. P. Mulholland (Exeter)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 796.

780:

Wagner, R. W. A note on linear difference equations. Amer. Math. Monthly **65** (1958), 351–353.

A matrix method given by Friedman [*Principles and techniques of applied mathematics*, Wiley, New York, 1956; MR **18**, 43] and Milne-Thomson [*The calculus of finite differences*, Macmillan, London, 1933] for solving homogeneous linear difference equations is extended to give a formal solution of the nonhomogeneous equation.

T. N. E. Greville (Kensington, Md.)

SEQUENCES, SERIES, SUMMABILITY

See also 788, 812.

781:

Zierler, Neal. Linear recurring sequences. J. Soc. Indust. Appl. Math. **7** (1959), 31–48.

The author considers homogeneous linear recurring sequences with coefficients in a finite (Galois) field K with $r+1$ elements. For discussion of length of a periodic sequence and number of periodic sequences satisfying a given linear recurrence, the paper is encyclopedic. Some of the less standard topics discussed are the following. Let $S=\{a_i\}_{-\infty}^{\infty}$ satisfy the (finite) recurrence relations $\sum_j c_j a_{k-j}=0$, $k=0, \pm 1, \cdots$, where the c_j are fixed ($c_0 c_n \neq 0$). The periods of $S_s=\{a_{us}\}$ are investigated, $u=is$, s fixed >1. Let S have period r; then S_s has period r if $(s, r)=1$.

Let η be a complex-valued function on K. The corresponding auto-correlation function $\theta(s)=\theta_{S,\eta}(s)$ is defined as $\sum \eta(a_i)\overline{\eta(a_i+s)}$, summation being extended over a period. When $r=1$, $\eta(0)=1$, $\eta(1)=-1$, this function is said to be important in applications.

J. L. Brenner (Menlo Park, Calif.)

782:

★Teghem, J. Sur des extensions d'une méthode de prolongement analytique de Borel. Colloque sur la théorie des suites, tenu à Bruxelles du 18 au 20 Décembre, 1957, pp. 87–95. Centre Belge de Recherches Mathématiques. Librairie Gauthier-Villars, Paris; Établissements Ceuterick, Louvain; 1958. 167 pp. 220 fr. belges.

This is an expository paper, giving a concise but clear survey of the cases to which Borel's principle of analytic continuation [E. Borel, J. Math. Pures Appl. **1896**, 441–451; Ann. Sci. École Norm. Sup. Paris **16** (1899), 9–131; see also Okada's theorem, (8.3, II) in the reviewer's *Infinite matrices and sequence spaces*, Macmillan, London, 1950; MR **12**, 694; p. 189 et seq., with references there given] has so far been extended. In Borel's original proof, the generating domain A of summability is star-like and contains $|z|<1$. Borel [in the second paper mentioned above], and Leau [C. R. Acad. Sci. Paris **127** (1898), 607–609] extend the principle to star-like A containing the origin, but not the whole unit circle. The author [Acad. Roy. Belg. Bull. Cl. Sci. (5) **35** (1949), 177–185; **37** (1951), 20; Bull. Soc. Roy. Sci. Liège **14** (1945), 366–376; Thèse, Univ. Libre, Bruxelles, 1946; C. R. du Congrès de l'A.F.A.S., 1955, Caen; MR **11**, 21; **12**, 813; **8**, 511; **10**, 112] considers a connex simply-connected A not containing the origin, and finds that Borel's principle applies only with restrictions on $f(z)$ at infinity.

The author also mentions various extensions due to P. Vermes [J. Analyse Math. **2** (1952), 160–177; Acta Sci. Math. Szeged **14** (1951), 23–38; MR **14**, 739; **13**, 27]: (i) when A is a curvilinear star-domain; (ii) when A is not connected and contains the origin; (iii) when the method is conservative but not regular, and the generalized sum of $\sum z^k$ in A is not $(1-z)^{-1}$; (iv) the failure of the principle when A is not simply-connected. *R. G. Cooke* (London)

783:

Francia, Giovanni. Un criterio di sommazione per le serie di potenze. Boll. Un. Mat. Ital. (3) **13** (1958), 372–382. (English summary)

Let $S(x)=\sum_{m=0}^{\infty} a_m x^m$ be convergent for every $x \in C=\{x \mid |x|<r>0\}$. For any complex number $h \neq 0$ and any integer $p \geq 0$, define: $a_m=0$ if $m=-1, -2, -3, \cdots$,

$$b_0(h, p)=a_0; \quad b_n(h, p)=\sum_{\lambda=1}^{n+p}\binom{n+p-1}{\lambda-1}a_{\lambda-p}h^{\lambda-p};$$

$$S_h^{(p)}(x)=[h/(h+x)]^p \sum_{n=0}^{\infty} b_n(h, p)[x/(h+x)]^n.$$

Then the series $S_h^{(p)}(x)$ converges in some region $\Gamma^{(h)}$ which is independent of the value of p and contains a neighborhood of the origin. If $x \in C \cap \Gamma^{(h)}$, both series $S(x)$ and $S_h^{(p)}(x)$ have the same sum, and the union of all the regions $\Gamma^{(h)}$ contains C. Particular cases are studied with detail and the application to numerical computation is illustrated with an example.

A. G. Azpeitia (Amherst, Mass.)

APPROXIMATIONS AND EXPANSIONS

See also 961, 962, 1120.

784:

Hsu, L. C. A refinement of the line integral approximation method and its application. Sci. Record (N.S.) **2** (1958), 193–196.

The problem of approximating a double or triple integral by a line integral, spiraling over (or scanning) the region of integration, has been investigated repeatedly [see J. Marechal, Amer. Opt. Soc. **37** (1947), 403–404; J. E. Wilkins, Bull. Amer. Math. Soc. **55** (1949), 191–192; MR **10**, 516; E. Grosswald, Proc. Amer. Math. Soc. **2** (1951), 706–709; MR **13**, 215; H. D. Brunk and G. M. Ewing, Proc. Amer. Math. Soc. **4** (1953), 287–295; MR **14**, 735; L. C. Hsu, Tohoku Math. J. (2) **9** (1957), 45–55; Acta Math. Sinica **3** (1953), 148–153; Bull. Calcutta Math. Soc. **48** (1956), 191–195; MR **19**, 884; **17**, 147]. Continuing his previous work, the author investigates the rapidity of convergence. Let $F(x, y)$ be continuous and have continuous partial derivatives in x up to the order m, let $\langle x \rangle = x-[x]$ stand for the fractional part of the real number x, denote by $B_r(y)$ the Bernoulli polynomial and by N a large integer. Then, using Euler's summation formula, the author proves

$$(*) \quad \int_0^1\int_0^1 F(x, y)dxdy = \int_0^1 F(x, \langle Nx \rangle)dx - \sum_{r=1}^{m-1}\frac{1}{r!}\left(\frac{1}{N}\right)^r \int_0^1 \left(\frac{\partial^{r-1}F}{\partial x^{r-1}}\right)_{x=0}^{x=1} B_r(y)dy + O(N^{-m})$$

and the formula obtained from (*) by passing to polar coordinates. *E. Grosswald* (Philadelphia, Pa.)

785:

Ziza, O. A. Some subsystems of orthogonal systems of functions. Dokl. Akad. Nauk SSSR **124** (1959), 257–259. (Russian)

An orthonormal system $\varphi_n(x)$ $(n=1, 2, \cdots)$ on $[a, b]$ is called a system of unconditional convergence if each series $\sum c_k \varphi_{n_k}(x)$ with $\sum c_k^2 < +\infty$ and the φ_{n_k} obtained from the φ_n by rearrangement is almost everywhere convergent. The main theorem is that each orthonormal system with $\sup |\varphi_n(x)| < +\infty$ a.e. contains a subsystem of unconditional convergence. *G. G. Lorentz* (Syracuse, N.Y.)

786:

Keogh, F. R. On Rademacher series with bounded sum. J. London Math. Soc. **33** (1958), 454–455.

Let $r_n(t)$, $n \geqq 0$, be the Rademacher functions. The author proves that if $f(t) = \sum a_n r_n(t)$ is essentially bounded, then $\sum |a_n| < \infty$.

{A much simpler proof than the author's is the following. Given $N > 0$, we can choose x so that $r_n(x) = \operatorname{sign} a_n$ for $n = 0, 1, \cdots, N-1$. Let α, β be defined by

$$p \cdot 2^{-N} = \alpha \leqq x < \beta = (p+1)2^{-N}.$$

Then

$$\sum_{n=0}^{N-1} |a_n| = \sum_{n=0}^{N-1} a_n r_n(x) = 2^N \int_\alpha^\beta f(t)dt \leqq M = \text{ess sup } f(t).\}$$

N. J. Fine (Princeton, N.J.)

787:

Vituškin, A. G. Best approximations of differentiable and analytic functions. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 418–420. (Russian)

Using the notion of the ε-entropy of subsets of a metric space by Kolmogorov [same Dokl. **108** (1956), 585–588; MR **18**, 324] and his own computations [ibid. **114** (1957), 686–689; **117** (1957), 745–747; MR **19**, 841; **20** #3750] the author obtains estimates from below for the maximum of the degree of uniform approximation by certain rational expressions for some functions defined on a set g of the r-dimensional complex space X_r. Let $g = I_r$ be an r-dimensional closed parallelepiped; let F be the set of all p times differentiable functions $f(x)$, bounded by c on I_r, with all derivatives of order p satisfying a Hölder condition with exponent $\alpha > 0$ and the constant $l > 0$. Let s, n, m be positive integers and let for all α_i, β_j with $\sum_1^n \alpha_i \leqq s$, $\sum_1^m \beta_j \leqq s$ functions $f_{\alpha_1, \cdots, \alpha_n}$, $g_{\beta_1, \cdots, \beta_m}$ be selected. Let $e_{s,n,m}(f, F)$ denote the infimum, taken for all a_i, b_j, of the uniform norm on I_r of

$$f(x) - \sum_{\alpha_i} \prod_{i=1}^{n} (a_i)^{\alpha_i} f_{\alpha_1 \cdots \alpha_n}(x) \Big/ \sum_{\beta_j} \prod_{j=1}^{m} (b_j)^{\beta_j} g_{\beta_1 \cdots \beta_m}(x),$$

and $e_{s,n,m}(F) = \max_{f \in F} e_{s,n,m}(f, F)$. Then

$$e_{s,n,m}(F) \geqq A[(n+m+1)\log(s+2)]^{-(p+\alpha)/r},$$

where A depends on p, α, l, c, I_r. A similar estimate is

$$e_{s,n,m}(F_{\gamma,g}) \geqq Aq[(n+m)\log_2(s+1)]^{1/n},$$

where $F_{\gamma,g}$ is the set of all analytic functions regular in the region g of X_r which have an analytic extension bounded by c into the region $\gamma \supset g$. The author remarks that the first estimate shows that approximation by ordinary polynomials is near to the best possible, while no such conclusion is implied by the second estimate.

G. G. Lorentz (Syracuse, N.Y.)

788:

Ganzburg, I. M.; and Timan, A. F. Linear processes of approximation by algebraic polynomials to functions satisfying a Lipschitz condition. Izv. Akad. Nauk SSSR. Ser. Mat. **22** (1958), 771–810. (Russian)

Let λ_{nk} $(k = 0, \cdots, n;\ n = 0, 1, \cdots)$ be arbitrary. For $f(x)$ continuous on $[-1, +1]$ we consider its Fourier series

$$(*) \qquad \sum_{k=0}^{\infty} C_k T_k(x)$$

with respect to the Čebyšev polynomials $T_k(x)$ and the weight $(1-x^2)^{-\frac{1}{2}}$ and put

$$u_n(f, x, \lambda) = \sum_{k=0}^{n} \lambda_{nk} c_k T_k(x).$$

A considerable part of the paper is devoted to a detailed exposition of the results of A. F. Timan [Dokl. Akad. Nauk SSSR **101** (1955), 221–224; MR **16**, 920] and Ganzburg [ibid. **116** (1957), 727–730; MR **19**, 1175]. In the remainder, estimates of

$$\mathscr{E}_n(\alpha, x, \lambda) = \sup_{f \in H^\alpha} |f(x) - u_n(f, x, \lambda)|$$

are given. Here H^α denotes the Lipschitz class with exponent $0 < \alpha \leqq 1$ and the constant 1. If u_n are the arithmetic means of the series,

$$(*) \qquad \mathscr{E}_n(\alpha, x, \lambda) = [C(\alpha) + o(1)](1-x^2)^{\alpha/2} n^{-\alpha} + \delta_n^\alpha(x),$$

where $C(\alpha) = 2\Gamma(\alpha)\sin\frac{1}{2}\alpha\pi/\pi(1-\alpha)$, $\delta_n^\alpha(x) = O(|x|^\alpha n^{-2\alpha})$ for $0 < \alpha < \frac{1}{2}$, $= O(|x|^\alpha n^{-1})$ for $\frac{1}{2} < \alpha < 1$ and $= O(|x|^{\frac{1}{2}} n^{-1}\log n)$ for $\alpha = \frac{1}{2}$. For more general λ the results are less precise. If the λ_{nk} are decreasing and convex in k,

$$\mathscr{E}_n(\alpha, x, \lambda) = C_\alpha(n)(1-x^2)^{\alpha/2} n^{-\alpha} + O(n^{-\alpha}) \quad (0 < \alpha < 1),$$

where $C_\alpha(n)$ is a product of $\sum_{k=0}^{n} \lambda_{nk}(n-k+1)^{-1}$ with a simple function of α. Similar results hold for $\alpha = 1$.

G. G. Lorentz (Syracuse, N.Y.)

789:

★Aljančić, S. Quelques cas particuliers de passage à la limite dans les développements asymptotiques. Colloque sur la théorie des suites, tenu à Bruxelles du 18 au 20 Décembre 1957, pp. 96–108. Centre Belge de Recherches Mathématiques. Librairie Gauthier-Villars, Paris; Établissements Ceuterick, Louvain; 1958. 167 pp. 220 fr. belges.

Suppose that $\{q_n(\lambda)\}$ is an ascending scale of functions which are positive definite for λ sufficiently large, and suppose that $\Phi_\lambda(t)$ admits of an asymptotic expansion

$$\Phi_\lambda(t) \sim \sum_n P_n(t)/q_n(\lambda) \quad (\lambda \to \infty).$$

Let ψ be a strictly monotone function and $\mathscr{L}$ an operator with domain the class of functions of bounded variation on any interval and such that

$$\mathscr{L}(f) = \int_0^\infty e^{i\tau t}\psi(t)df(t).$$

The author determines necessary and sufficient conditions in order that

$$\mathscr{L}(\Phi_\lambda) \sim \sum_n \mathscr{L}(P_n)/q_n(\lambda) \quad (\lambda \to \infty).$$

N. D. Kazarinoff (Ann Arbor, Mich.)

790:

Perron, Oskar. Über einen Kettenbruch von Ramanujan. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1958**, 19–23.

The author derives the value of the Ramanujan continued fraction

$$1 + b + \cfrac{qx}{1+qb} + \cfrac{q^2x}{1+q^2b} + \cfrac{q^3x}{1+q^3b} + \cdots,$$

namely, $(1+b)P(b,x)/P(qb,qx)$, $|q|\neq 1$, where

$$P(b,x) = 1+\sum_{p=1}^{\infty}\frac{q^{p^2}}{(1-q)(1-q^2)\cdots(1-q^p)} \times\frac{x^p}{(1+b)(1+qb)\cdots(1+q^{p-1}b)}.$$

{Reviewer's note: This can be obtained as a limit case of the Heine continued fraction [cf. Perron, *Die Lehre von den Kettenbrüchen*, 3rd ed., vol. II, Teubner, Stuttgart, 1957; MR **19**, 25; p. 125]. Professor Perron (cf. p. 127) applies the same limit technique to a special case of the Ramanujan continued fraction considered above.}

E. Frank (Chicago, Ill.)

FOURIER ANALYSIS

See also 819, 820.

791:

Salem, Raphaël. Recherches récentes sur l'unicité du developpement trigonométrique. Enseignement Math. (2) **4** (1958), 284–291.

A set $E\subset(0,2\pi)$ is a set of uniqueness (U) if the only trigonometric series $\sum_0^\infty a_n\cos nx+b_n\sin nx$ that converges to zero everywhere outside E is the trivial series $a_n\equiv b_n\equiv 0$. Otherwise, E is a set of multiplicity (M). In this expository article, a survey is made of progress on the problem of classifying sets E as U or M. Many intrinsically interesting questions in Diophantine approximation arise. The completely solved case where E is a Cantor set with constant ratio of dissection illustrates the strong role that arithmetic properties can play.

L. A. Rubel (Princeton, N.J.)

792:

Petrov, I. M. Order of approximation of functions of class Z by certain polynomial operators. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 127–131. (Russian)

The exact values of $\lim_{n\to\infty}(nb_n)$ are computed, where $b_n=\max_{f\in Z,\,|x|\leq\pi}|f(x)-K_n f(x)|$, Z is the class of 2π-periodic functions with

$$|f(x+h)-2f(x)+f(x-h)|\leq 2h$$

and $K_n f$ is one of two special singular trigonometric integrals (one of these is the integral of Jackson). In both cases the maximum is attained for $f(x)=|x|$.

G. G. Lorentz (Syracuse, N.Y.)

793:

Korneĭčuk, N. P. Approximation of periodic functions satisfying Lipschitz's condition by Bernstein-Rogosinski's sums. Dokl. Akad. Nauk SSSR **125** (1959), 258–261. (Russian)

The author estimates

$$E_n(\alpha) = \sup_{\varphi}\max|f(x)-B_n(f,x)|,$$

where $B_n(f,x)$ is Rogosinski's expression $\frac{1}{2}S_n(f,x+h_n)+\frac{1}{2}S_n(f,x-h_n)$, $h_n=\pi/(2n+1)$, $S_n(f,x)$ is the nth partial sum of the Fourier series of f and the supremum is taken for all f of the Lipschitz class of order α, $0<\alpha\leq 1$ and with constant 1. He obtains, for example,

$$E_n(1) = \pi/(2n+1)+\varepsilon_n/(n+1)+o(n^{-1})$$

with $0<\varepsilon_n<0.06$. *G. G. Lorentz* (Syracuse, N.Y.)

794:

Kahane, J. P.; et Salem, R. Sur la convolution d'une infinité de distributions de Bernoulli. Colloq. Math. **6** (1958), 193–202.

Let $\sum_0^\infty r_k$ be a convergent series of positive terms with sum one, and let $\{t_k\}$ be defined by $r_k=t_0t_1\cdots t_{k-1}(1-t_k)$. The function α defined by the $a(u)=\prod_0^\infty\cos\pi ur_k$ is the characteristic function of a positive measure with support $E\subset[-\pi,\pi]$. Jessen and Wintner had shown [Trans. Amer. Math. Soc. **38** (1935), 48–88] that the measure μ was either absolutely continuous or singular. The authors set up a measure on the space of sequences and show, for almost all sequences $\{t_k\}$ such that $t_k>\frac{1}{2}$, $k=1,2,\cdots$, that μ is absolutely continuous. (In this case $E=[-\pi,\pi]$.) An explicit determination is also made for μ. Particular attention is given to the case $r_k=t^k$.

P. Civin (Eugene, Ore.)

795:

Mandelbrojt, S. Le problème de Watson généralisé et ses applications. Ann. Acad. Sci. Fenn. Ser. A. I. no. **251/6** (1958), 11 pp.

The author gives several new applications of the theory developed in his book [*Séries adhérentes, regularisation des suites, applications*, Gauthier-Villars, Paris, 1952; MR **14**, 542]. For example, let $f\in L(-\infty,\infty)$. The spectrum of f, $\sigma(f)$, is the carrier of the Fourier transform of f. If $f\in C^\infty$ and $\|f^{(n)}\|\leq M_n$ (L-norm) then we say that $f\in L\{M_n\}$. For a closed set E let $E_a=\bigcup[x-a,x+a]$ $(x\in E)$. For $c>0$, we say that the set E and the sequence of positive numbers $\{M_n\}$ are associated A_c if the conditions $f\in L\{M_n\}$, $\sigma_f\subset E_c$, imply that all translates of f are limits (in L) of finite linear combinations of the $f^{(n)}$. Let $c>0$ be such that $E_c\neq(-\infty,\infty)$. We say that the set E and the sequence $\{M_n\}$ are associated U_c if the only function $\phi(z)$ analytic in $C\backslash E_c$ (C denotes the complex plane, $\backslash$ denotes set-theoretic difference) and such that $|\phi(z)z^ny|\leq M_n$ $(n\geq 1)$ is the identically zero function.

Theorem. If $0\in E$ and E and $\{M_n\}$ are associated A_c then they are also associated $U_{c'}$ for all c' $(0<c'<c)$. Conversely, if they are associated U_c then they are also associated $A_{c'}$ for $0<c'<c$. The proofs are outlined.

A. Shields (Ann Arbor, Mich.)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

See 726, 803, 804.

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 769.

796:

Ghermanescu, Michel. Équations intégrales aux deux limites variables. C. R. Acad. Sci. Paris **248** (1959), 1104–1105.

The author considers the equation

$$\varphi(x)-\lambda\int_{\theta(x)}^{x}K(x,s)\varphi(s)ds = 0, \tag{1}$$

where φ is to be found. Assuming that θ is n-periodic (that is, with $\theta_1=\theta$, $\theta_k=\theta(\theta_{k-1})$, we have $\theta_n(x)=x$), he proves that every solution of (1) is a solution of

$$(2)\quad \varphi(x)-\lambda\int_0^x [K(x,s)-\alpha_k^{n-1}K(\theta_{n-1}(x),s)]\varphi(s)ds=\Phi_k(x),$$

where $\Phi_k(x)$ satisfies

$$(3)\quad \Phi_k(x)+\alpha_k\Phi_k(\theta)+\cdots+\alpha_k^{n-1}\Phi_k(\theta_{n-1})=0,\quad \alpha_k^n=1.$$

Conversely, solutions of (2) and (3) are solutions of (1). The general solution of (3) was given earlier [same C. R. **243** (1956), 1593–1595; **244** (1957), 543–544; MR **18**, 581].

R. P. Boas, Jr. (Evanston, Ill.)

797:

Przeworska-Rolewicz, D. Étude des systèmes d'équations intégrales singulières pour les arcs non fermés par la méthode des approximations successives. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 697–703.

The author considers systems of integral equations of the form

$$u(t)=\lambda\int_L \frac{K[t,\tau u(\tau)]}{t-\tau}\,d\tau=\lambda\hat{K}u,$$

L denoting open arcs in the complex plane. An appropriate Banach space is introduced with norm designed to admit the possibility of singular behavior at the end points of the arcs of L. The principal result is that with suitable restrictions on K the operator $\hat{K}u$ satisfies a Lipschitz condition. It follows that equation (*) has a unique solution for suitably small λ.

R. C. MacCamy (Pittsburgh, Pa.)

798:

Evgrafov, M. A. On the asymptotic behaviour of solutions of simultaneous linear equations. Dokl. Akad. Nauk SSSR **121** (1958), 407–410. (Russian).

Let $P_n(z)=\sum_{-n+1}^{\infty}a_{mn}z^m$, $0<|z|<d_n$ $(n=1,\ 2,\ \cdots)$. Suppose that (1) there exist two sequences $\{\rho_n\}$ and $\{R_n\}$, $\rho_n<R_n$ such that

$$\lim_{n\to\infty}\max_{|z|=1}\left|\frac{P_{n+1}(z/R_{n+1})}{P_n(z/R_n)}\right|=\lim_{n\to\infty}\max_{|z|=1}\left|\frac{P_{n+1}(z/\rho_{n+1})}{P_n(z/\rho_n)}\right|=1;$$

(2) there exists $\{r_n\}$ such that $\rho_n<r_n<R_n$, $P_n(z)\neq 0$ for $|z|=r_n$, and the net change of $\arg P_n(z)$ on the circle $|z|=r_n$ is zero; (3) for sufficiently large n, $P_n(z)$ has k_1 zeros $\{\lambda_i^{(n)}\}_{-1}^{-k_1}$ in the annulus $r_n\leq|z|\leq R_n$ and k_2 zeros $\{\lambda_i^{(n)}\}_1^{k_2}$ in the annulus $\rho_n\leq|z|\leq r_n$ such that $\lambda_i^{(n)}\neq\lambda_j^{(n)}$, $i\neq j$; (4) if $P_{n,m}=P_n/(z-\lambda_m^{(n)})$, then

$$\lim_{n\to\infty}\frac{P_{n+1,i}(\lambda_i^{(n)})}{P_{n+1,i}(\lambda_i^{(n+1)})}=1,\quad \sum_1^{\infty}|\beta_{ij}^{(n+1)}-\beta_{ij}^{(n)}|<\infty,$$

$$\sum_1^{\infty}|\gamma_{ij}^{(n)}|<\infty\quad (i\neq j),$$

where

$$\beta_{ij}^{(n)}=\frac{P_{n+1,i}(\lambda_j^{(n)})}{\lambda_j^{(n)}P_{n+1,i}(\lambda_j^{(n)})},\quad \gamma_{ij}^{(n)}=\lambda_j^{(n)}\beta_{ij}^{(n)}\beta_{ji}^{(n)}.$$

Let A be the matrix $(\alpha_{ij})_1^{\infty}$, $\alpha_{ij}=a_{i-j,\,i}$, and let $Y=\{y_1,y_2,\cdots\}$ be a solution of the system $AY=F$, $F=\{f_1,f_2,\cdots\}$, $f_n=0$ for $n>n_1$. The author's main theorem gives the asymptotic behavior of the y_n for large n under the additional assumption that

$$y_n=O[(1-\varepsilon)^n R_1\cdots R_n]$$

for ε positive and sufficiently small.

N. D. Kazarinoff (Ann Arbor, Mich.)

FUNCTIONAL ANALYSIS

See also 775.

799:

Morse, Marston; and Transue, William. The local characterization of vector function spaces with duals of integral type. J. Analyse Math. **6** (1958), 225–260.

This paper studies relationships between the global and local aspects of MT-spaces [same **4** (1954/55), 149–186; MR **17**, 469; J. Math. Pures Appl. **37** (1958), 343–363; MR **20** #6022]. We use the notation and terminology of the latter review, with the difference that in defining an MT-space A, one admits the possibility of a trivial (identically vanishing) seminorm $\mathcal{N}^A$.

Suppose A is a vector subspace of C^E which is C-proper (i.e. $x\in A$ entails $|x|\in A$, $\bar{x}\in A$); A is sectionally admissible if, for each compact $K\subset E$ and $x\in A$, $x\varphi_K\in A$ (φ_K = characteristic function of K). If further the vector subspace $A|K$ of C^K, formed of restrictions $x|K$ $(x\in A)$, equipped with the induced seminorm $\mathcal{N}^{A|K}(x)=\mathcal{N}^A(x\varphi_K)$, is an MT-subspace of C^K, A is said to be locally of MT-type (or to be a local MT-space). An MT-space is a local MT-space, provided it is sectionally admissible (but otherwise not necessarily so). Not every local MT-space is an MT-space, and Part I of the paper is concerned with the problem of finding global conditions on a local MT-space A which are necessary and sufficient in order that A be an MT-space. This is dealt with in Theorems 3.2 and 3.3, and a solution is provided by Corollary 3.1: A local MT-space A is an MT-space, if and only if $\int^*|x|d|\alpha|<+\infty$ for each $x\in A$ and each α in the measure dual of A, and $\lim_K x\varphi_K=x$ in the sense of A (as the variable compact K expands). On the other hand (Theorem 4.2), if A is an MT-space of Cauchy type, then each $A|K$ (seminormed by $\mathcal{N}^{A|K}$) has the same property; in particular, A is a local MT-space.

Part II of the paper deals with the problem of piecing together a consistent system of MT-sections $J(K)$ into an MT-space. More precisely, it is assumed that for each compact K, $J(K)$ is a sectionally admissible MT-subspace of C^K with seminorm $\mathcal{N}^{J(K)}$; the consistency means that if $K\subset H$, then $J(H)|K=J(K)$ as MT-spaces, i.e. as vector spaces with seminorms $\mathcal{N}^{J(H)}(x\varphi_K{}^H)$ and $\mathcal{N}^{J(K)}(x|K)$ which agree, for $x\in J(H)$, $\varphi_K{}^H$ being the characteristic function of K relative to H. Given such a system, $J^e(K)$ is defined to be the set of extensions $y^e(=y$ on K, $=0$ on $E-K)$ of $y\in J(K)$, seminormed by $\mathcal{N}^{J^e(K)}(y^e)=\mathcal{N}^{J(K)}(y)$. $J^e(K)$ is a seminormed vector subspace of C^E, easily seen to be sectionally admissible. J, the union of the $J^e(K)$ with K varying, is seminormed by $\mathcal{N}^J(x)=\mathcal{N}^{J^e(K)}(x)$ for any K for which $J^e(K)$ contains x. J is C-proper and sectionally admissible. The main result (Theorem 5.1) asserts that J, equipped with $\mathcal{N}^J$, is a sectionally admissible MT-subspace of C^E such that $J|K=J(K)$ as MT-spaces for each compact K.

Starting from the same system of $J(K)$ one can form $\bar{J}$ and $\bar{J}(K)$, the closures in $\mathscr{F}^J$ and $\mathscr{F}^{J(K)}$ respectively of J and $J(K)$. It then appears (Theorem 5.2) that $\bar{J}$ is sectionally admissible and $\bar{J}|K=\bar{J}(K)$ for each K; moreover, $\bar{J}$ is maximal and consists of those $x\in\mathscr{F}^J$ such that $x|K\in\bar{J}(K)$ for each K, whilst $\lim_K x\varphi_K=x$ in the sense of $\mathscr{F}^J$. One can also introduce a third space $\tilde{J}$ by defining a new seminorm

$$M^J(x)=\operatorname*{Sup}_K \mathscr{N}^{J(K)}(x|K)\leqq\mathscr{N}^J(x)$$

for $x\in C^E$ ($\mathscr{N}^{J(K)}$ being the natural extension of $\mathscr{N}^{J(K)}$ to C^K); M^J and $\mathscr{N}^J$ are shown to agree on $\mathscr{F}^J$; $\tilde{J}$ is then the set of those $x\in C^E$ such that $M^J(x)<+\infty$ and which are adherent to $\mathscr{K}_C$ in the sense of M^J. Roughly speaking, $\tilde{J}$ differs from $\bar{J}$ in much the same way as essentially integrable functions differ from integrable functions: $\tilde{J}\supset\bar{J}$, and a function in $\tilde{J}$ which vanishes outside a σ-compact set necessarily belongs to $\bar{J}$. If we define $\{M^J\}$ to consist of those $x\subset C^E$ which are locally α-negligible for each α in the measure dual of J, then (Theorems 6.1 and 6.2) $\tilde{J}=\bar{J}+\{M^J\}$ (direct sum if and only if $\{M^J\}=\{0\}$) and $\tilde{J}=J_\sigma+\{M^J\}$, where J_σ is the set of functions in J vanishing outside σ-compact sets. Various direct sum decompositions of the type $\tilde{J}=\bar{J}\oplus N$ or $\tilde{J}=J_\sigma\oplus N$, where N is one of several vector spaces of functions which are locally negligible for the measure dual of J, are derived (Corollaries 6.2, 6.3, 6.4).

{There are a few trivial misprints: p. 248, line 7 up: for "$K\supset H$", read "$K\subset H$"; p. 257, line 12 up: for "$\bar{J}$", read "$\tilde{J}$"; p. 257, lines 7 and 5 up: for "$\bar{J}$", read "$\tilde{J}$"; p. 258, line 10: for "–", read "$\bar{J}$".}

R. E. Edwards (London)

800:

Morse, M.; and Transue, W. The existence of vector function spaces with duals of integral type. Colloq. Math. **6** (1958), 95–117.

This paper continues the study of so-called MT-spaces [references in the preceding review]. The existence of an abundance of MT-spaces is illustrated by three processes: (1) a logical characterization of MT-spaces (§ 2); (2) the association of an MT-space A_ω with each of a certain class of sets ω of complex Radon measures (§ 3); (3) the generation of new MT-spaces from given ones by taking intersections and, where possible, unions (§§ 4, 5, 6). Briefly, an MT-space A is (I) a seminormed vector subspace of C^E, stable under $x\to|x|$ and $x\to\tilde{x}$, and containing $\mathscr{K}_C$ as a dense subset; to each η in A' (dual of A) corresponds then a measure $\hat{\eta}=\eta|\mathscr{K}_C$ and $\eta\to\hat{\eta}$ is an isomorphism of A' onto a vector space $\mathscr{A}'$ of measures, called the measure dual of A. It is assumed (II) that each x in A is integrable for each α in $\mathscr{A}'$ and that $\eta(x)=\int x\,d\hat{\eta}$ ($x\in A$, $\eta\in A'$).

A typical result of category (1) is (Theorem 2.2) that, if A satisfies (I), a necessary and sufficient condition for A to be an MT-space is that for each α in $\mathscr{A}'$ there exists a number $m_\alpha\geqq 0$ for which $N_C{}^1(x,\alpha)\equiv\int^*|x|\,d|\alpha|\leqq m_\alpha\cdot\mathscr{N}^A(x)$ for all x in A. Turning to process (2), start from a set ω of measures and define the saturated set ω^* as consisting of all measures β for which $|\beta(v)|\leqq\rho_\omega(v)\equiv\operatorname{Sup}_{\alpha\in\omega}\int|v|\,d|\alpha|$ for all v in $\mathscr{K}_C$. Then (Theorem 3.1), if ρ_ω is finite-valued, there exists a unique maximal MT-space A_ω such that

$$\mathscr{A}_\omega{}^u\equiv\{\alpha\in\mathscr{A}':|\textstyle\int v\,d\alpha|\leqq\mathscr{N}^{A_\omega}(v)\text{ for all }v\text{ in }\mathscr{K}_C\}$$

is identical with ω^*; and for this A_ω one has $\mathscr{N}^{A_\omega}|\mathscr{K}_C=\rho_\omega$. There is a trivial converse of this assertion: for any MT-space A, $\omega=\mathscr{A}'^u$ is saturated and ρ_ω is finite-valued.

Processes of category (3) are more complicated to describe in detail. They depend on a study of the lattice Λ of seminorms on $\mathscr{K}_C$. Roughly speaking, the intersection $J=\bigcap_{i\in I}A_i$ of a family of MT-spaces A_i may be made into an MT-space with the seminorm

$$\operatorname*{Inf}_{i\in I}[\mathscr{N}^{A_i}|\mathscr{K}_C]\quad(\text{Inf in }\Lambda),$$

first extended in a natural way to C^E and then restricted to J; and $\mathscr{J}'^u=\bigcap_{i\in I}\mathscr{A}_i{}^u$ (Theorem 5.1). Under certain conditions, J can also be considered as an MT-space with seminorm

$$\operatorname*{Sup}_{i\in I}[\mathscr{N}^{A_i}|\mathscr{K}_C];$$

and then $\mathscr{J}'^u=(\bigcup_{i\in I}\mathscr{A}_i{}'^u)^*$ (Theorem 6.1).

R. E. Edwards (London)

801:

Schäffer, J. J. A geometrical property of normed vector spaces. Bol. Fac. Ingen. Agrimens. Montevideo **6** (1957/58), 337–344. Also published as Fac. Ingen. Montevideo. Publ. Inst. Mat. Estadist. **3** (1958), 91–98. (Spanish. English summary)

The author considers the following problem for a Banach space of dimension >1. Let V represent the unit ball, $V=\{x:\|x\|\leqq 1\}$, and let m be a positive integer, k a positive constant. Do there exist functions h_i, $i=1,\cdots,m$, uniformly continuous in V with values in the space such that $x\in V$ implies $x=h_1(x)+\cdots+h_m(x)$ and $\|h_i(x)\|=k$? He proves that this problem may be solved for $m=4$ and any $k>\frac{1}{4}$. He proves that the problem cannot be solved for $m=2$, in that if $x\in V$ implies $x=h_1(x)+h_2(x)$, $\|h_i(x)\|=k$, then the h_i are discontinuous at the origin. In case of a Hilbert space, the solution (valid also for $k=\frac{1}{4}$) is given explicitly. An immediate application is: Let X be a normed space, let S be a topological space; let $C(S,X)$ be the normed space of all bounded continuous mappings of S into X with the supremum norm. Then for any $r>0$, the set S_r of functions of constant norm, $\|f(t)\|=r$ for all $t\in S$, contains a linear base of $C(S,X)$. The proofs are elementary.

E. R. Lorch (New York, N.Y.)

802:

Alexiewicz, A. On certain "weak" properties of vector-valued functions. Studia Math. **17** (1958), 65–68.

A theorem of B. J. Pettis [Duke Math. J. **5** (1939), 254–269; p. 257] asserts that a function $x(t)$ on a measure space to a (B)-space X is measurable-Bochner if and only if it is almost separably valued and, for every γ belonging to a norming set Γ (= determining manifold), the function $\gamma[x(t)]$ is measurable-Lebesgue. The author replaces "norming" by the weaker condition "total". Further, he shows that $x(t)$ is a Baire function if it is separably valued and $\gamma[x(t)]$ is a Baire function for every γ in a total set. He also shows that, in Dunford's theorem [Trans. Amer. Math. Soc. **44** (1938), 305–356; p. 354] on holomorphic vector-valued functions, it is possible to replace "norming" sets by "total".

E. Hille (New Haven, Conn.)

803:

Zaidman, S. La représentation des fonctions vectorielles par des intégrales de Laplace-Stieltjes. Ann. of Math. (2) **68** (1958), 260–277.

Let $f(s)$, $s>0$, be an infinitely differentiable scalar function; define the operator:

$$L_{k,t}[f] = [(-1)^k/k!](k/t)^{k+1}f^{(k)}(k/t).$$

Then Widder's theorem on the representation of a function by a Laplace-Stieltjes integral may be given the following form: The necessary and sufficient condition that $f(s)=\int_0^\infty e^{-st}d\alpha(t)$, where $\alpha(t)$ is of bounded variation on $I^+=[0,\infty]$, is that (1) $f(s)$ be infinitely differentiable for $s>0$; (2) there exists an $M>0$ such that $\int_0^\infty |L_{k,t}[f]|dt<M$, $k=1, 2, \cdots$. The author considers this same problem in case the function f takes its values in a Banach space X, a situation considered by Hille [*Functional Analysis and semi-groups*, Amer. Math. Soc., New York, 1948; MR **9**, 594; Chapt. X]. He gives four theorems, each with necessary and sufficient conditions for such a representation. The first two apply to the case that X is reflexive (in particular, theorem 2 solves the problem proposed by Hille). In the remaining theorems, X is arbitrary. For the sake of illustration we state the first theorem: If X is reflexive, then the following conditions are necessary and sufficient for the representation of $f(s)$ by a Laplace-Stieltjes integral: (1) $f(s)$ is infinitely differentiable in the strong topology of X; (2) for $k=1, 2, \cdots$, the functions $\alpha_k(t)=\int_{0+}^t L_{k,u}[f]du$ exist as strong limits of the integrals $\int_\epsilon^t$ and the set V_2 to be presently defined is bounded in X. Here V_2 is the set of all elements in X of the form $\sum_{i=1}^n \varepsilon_i[\alpha_k(t_{i+1})-\alpha_k(t_i)]$, $n=1, 2, \cdots$, $k=1, 2, \cdots$, $\varepsilon_i=\pm 1$, and $t_1<t_2<\cdots<t_{n+1}$ are arbitrary in I^+. In his development, the author examines several types of functions of bounded variation (for short: b.v.), $\alpha(t)$, with values in X. The class $V_f{}^{I^+}$ is that of the functions of weak b.v.; the class $V_r{}^{I^+}$ is that of the functions of strong b.v., defined with the help of an obviously interpreted inequality $\sum_{i=1}^n \|\alpha(t_{i+1})-\alpha(t_i)\|<M$; finally, the class $V_\sigma{}^{I^+}$ consists of all functions $\alpha(t)$ for which the elements of the form $\sum_{i=1}^n \varepsilon_i[\alpha(t_{i+1})-\alpha(t_i)]$ form a compact set in X. The theorems are preceded by ten lemmata, devoted for the most part to the proof of the existence of various strong limits needed for the final assault.

E. R. Lorch (New York, N.Y.)

804:

Miller, J. B. A continuum of Hilbert spaces in L^2. Proc. London Math. Soc. (3) **9** (1959), 208–226.

The space $G_\lambda{}^p$ ($\lambda>0$) consists of those $f(x)$ such that

$$(1) \qquad f(x) = \Gamma(\lambda)^{-1}\int_x^\infty (t-x)^{\lambda-1}f^{(\omega)}(t)dt \quad (x>0),$$

and

$$(2) \qquad t^\omega f^{(\omega)}(t) \in L^p(0,\infty).$$

For $\lambda=0$ let $G_0{}^p=L^p$. These spaces are investigated with respect to continuity, orders of magnitude and Lipschitz conditions. It is shown that with appropriate norm $G_\lambda{}^2$ is a Hilbert space and questions of convergence are discussed. A class of closed, self-adjoint, unitary transformations in $G_\lambda{}^2$ is described as well as a method for finding O.N. sets from corresponding O.N. sets in L^2.

J. Blackman (Syracuse, N.Y.)

805:

Brainerd, Barron. On a class of lattice-ordered-rings. II. Nederl. Akad. Wetensch. Proc. Ser. A **60**=Indag. Math. **19** (1957), 541–547.

An F-ring is a σ-complete vector lattice that is an algebra with identity element 1 in which the product of positive elements is positive, and 1 is a weak order unit; cf. the author's earlier paper [Proc. Amer. Math. Soc. **8** (1957), 673–684; MR **19**, 629]. The main result of this paper is the theorem that every regular (in the sense of von Neumann) F-ring is a homomorphic image of a ring of measurable functions under a homomorphism that preserves countable lattice operations.

Also, a one-one correspondence between the class of F-rings and the class of σ-complete Boolean algebras is exhibited. Finally, by exploiting some results of Nakano [*Modern spectral theory*, Maruzen, Tokyo, 1950; MR **12**, 1951], the author obtains spectral integral decompositions for the elements of a larger class of archimedian lattice-ordered algebras.

M. Henriksen (Lafayette, Ind.)

806:

Brainerd, Barron. F-rings of continuous functions. I. Canad. J. Math. **11** (1959), 80–86.

An F-ring is a σ-complete vector lattice that is an algebra in which the product of positive elements is positive, and which has an identity that is a weak order unit. The author characterizes the algebra $C(X)$ of all continuous real-valued functions on a P-space X among the class of F-rings. (X is a P-space if every G_δ is open; see Trans. Amer. Math. Soc. **77** (1954), 340–362 [MR **16**, 156].) A simpler characterization is obtained in case X is a P-space that is a Lindelöf space. (The author uses "upper $\aleph_1$-compact" for the latter.) The characterizations are internal in each case.

M. Henriksen (Lafayette, Ind.)

807:

Banaschewski, Bernhard. On certain extensions of function rings. Canad. J. Math. **11** (1959), 87–96.

Here a function ring A is a Banach algebra isomorphic to the ring of bounded continuous functions on a topological space E. A function ring B containing A is a normal extension if it is not directly decomposable, and if there is a group G of automorphisms of B for which A constitutes the set of invariant elements. If G is unique [and finite] the extension is called Galois [and finite]. The restriction-to-A map applied to the space $\mathscr{B}$ of complex homomorphisms of B displays $\mathscr{B}$ as a fibre space over $\mathscr{A}$, the corresponding space for A. The algebraic concepts defined above are then appropriately translated into topological concepts relating $\mathscr{B}$ and $\mathscr{A}$. Most of the paper consists of topological results on relevant types of fibre spaces, which in turn provide a characterization of the finite-Galois situation. This cannot be easily described here. A simpler result, translated back into algebra terms, is that if A is not directly decomposable, and G is given, then a normal extension B exists, having G as the associated group.

R. Arens (Los Angeles, Calif.)

808:

Kantorovič, L. V.; and Rubinšteĭn, G. Š. On a space of completely additive functions. Vestnik Leningrad. Univ. **13** (1958), no. 7, 52–59. (Russian. English summary)

Details of results announced in Dokl. Akad. Nauk SSSR **115** (1957), 1058–1061 [MR **20** #1219].

M. Jerison (Lafayette, Ind.)

809:

Cotlar, Mischa; and Panzone, Rafael. On almost orthogonal operators in L^p-spaces. Acta Sci. Math. Szeged **19** (1958), 165–171.

Let $K_1(x), \cdots, K_N(x)$ be integrable functions defined on euclidean n-space E^n, and let $T_1 f = f * K_1 = \int_{E^n} f(y)K_1(x-y)dy$, $K(x) = K_1(x) + \cdots + K_N(x)$, $Tf = f * K$, $K_{ij} = K_i * K_{i+j}$; let $\|f\|_p$ denote the usual L^p norm. A theorem of Cotlar [Rev. Mat. Cuyana **1** (1955), 41–55; MR **18**, 219] is the following: If the kernels K_{ij} satisfy $\|K_{ij}\|_1 \leq c\varepsilon^j$, where $0 \leq \varepsilon < 1$, and if $f \in L^2(E^n)$, then $\|f * K\|_2 \leq c_1\|f\|_2$, where $c_1 = c_1(\varepsilon, c)$ and does not depend on N. The operators T_i are on L^2 with $\|T_i\| \leq \|K_i\|_1$ and furthermore $\|T_i T_{i+j}\| \leq c\varepsilon^j$. For this reason they are called almost orthogonal on L^2. In the present paper, the authors prove three theorems generalizing these results to L^p spaces and to subspaces of E^n. We state the least complex of these: Let $0 \leq \gamma < n$. Let K_{ij} satisfy $\int_{E^n} |K_{ij}(x+h) - K_{ij}(x)|dx \leq c\varepsilon^j |h|^{2\gamma}$, $1 \leq i \leq i+j \leq N$, for $0 \leq \varepsilon < 1$ and for every $h \in E^n$; if $p = 2n/(n+\gamma)$, $1/p + 1/q = 1$, then $\|f * K\|_q \leq c_1\|f\|_p$ holds for every $f \in L^p(E^n)$, where c_1 depends on ε, γ, and c only. For $\gamma = 0$ (and hence $p = 2$), this gives Cotlar's theorem. The proofs are secured with the help of auxiliary results based on Fourier transform theory and various inequalities (Hausdorff-Young, Hardy-Littlewood-Paley; also Pitt). Applications are indicated and will be published in full elsewhere.

E. R. Lorch (New York, N.Y.)

810:

Raimi, Ralph A. Mean values and Banach limits. Proc. Amer. Math. Soc. **8** (1957), 1029–1036.

Let E be a closed subspace of the Banach space of all real-valued bounded Lebesgue-measurable functions on the reals which contains all bounded uniformly continuous functions and is translation-invariant (if $f \in E$, $s \in R$, then $f_s \in E$, where $f_s(x) = f(x+s)$). A functional $\phi \in E'$ is called a Banach limit if: (a) $\|\phi\| = 1$; (b) $\phi(e) = 1$; (c) $\phi(f_s) = \phi(f)$. For certain $f \in E$, $\lim 2\alpha^{-1} \int_{-\alpha}^{\alpha} f(x+t)dt$ exists uniformly in x. This limit is a constant $m(f)$ which is a translation-invariant positive functional of norm 1. The author discusses properties of norm-preserving extensions of the functional $m(f)$. In particular, the extreme values of these extensions are explored. Furthermore, all extensions of $m(f)$ are Banach limits, and if E is the space of bounded uniformly continuous real functions on the reals, the set of extensions of $m(f)$ is exactly the set of Banach limits.

L. Brown (Detroit, Mich.)

811:

Raimi, Ralph A. On a theorem of E. Følner. Math. Scand. **6** (1958), 47–49.

Let G be a group and L a linear space of bounded real-valued functions on G with the norm $\|f\| = \sup\{|f(x)| \mid x \in G\}$, such that $\sup(f, g) \in L$ if f, $g \in L$. Let $\mathfrak{F} = \{T_a \mid a \in G\}$ where, for each $a \in G$, T_a is the operator defined by the equation $T_a f(x) = f(xa)$. Denote with N the closed linear space spanned by $\{Tf - f \mid f \in L,\ T \in \mathfrak{F}\}$. A linear continuous form on L, x', is a right Banach mean if $x' \not\equiv 0$, $x'(f) \geq 0$ for $f \geq 0$ and $x'(f) = 0$ for $f \in N$. The author gives in this note a new proof of the following result [see E. Følner, Math. Scand. **5** (1957), 5–11; MR **20** #207]: If L has no right Banach mean then $L = N$.

C. T. Ionescu Tulcea (New Haven, Conn.)

812:

Erdös, P.; and Piranian, G. The topologization of a sequence space by Toeplitz matrices. Michigan Math. J. **5** (1958), 139–148.

The purpose of this paper is to show how the convergence fields of certain regular Toeplitz matrices can serve as neighbourhoods in the topologization of a factor space of the space m of bounded sequences.

The following notation and terminology are employed. $A = (a_{n,k})$ is a Toeplitz matrix, $x = \{x_k\} \equiv \{x(k)\}$ is a sequence of complex constants, Ax is the sequence $\sum_{k=0}^{\infty} a_{n,k}x_k$ $(n = 0, 1, 2, \cdots)$; if Ax converges to a, we write $Ax \to a$, and we say that A evaluates x to a. The statement $x \sim y$ means that there exists a constant $\lambda \neq 0$ and a convergent sequence c such that $y_k = \lambda x_k + c_k$. The relation $y \sim x$ is an equivalence relation; m/L denotes the space of equivalence classes which it determines in m, and X represents the equivalence class to which x belongs. If $x \sim y$, then $Ax \sim Ay$ for every regular T-matrix A; AX thus denotes the equivalence class to which Ax belongs. The set of bounded sequences evaluated by A is called the convergence field of A in m (or the bounded convergence field of A), and by immediate extension, we can speak of the convergence field of A in m/L. If $\{k_r\}$ is an increasing sequence of positive integers, we say that $\{\xi_k\}$ wanders slowly over $\{k_r\}$ if

$$\lim_{r \to \infty} \max_{k_r < k \leq k_{r+1}} |\xi(k) - \xi(k_r)| = 0.$$

If ξ wanders slowly over $\{k_r\}$ and η wanders slowly over $\{n_r\}$, and if $\xi(k_r) - \eta(n_r) \to 0$, we say that ξ and η are $\{k_r, n_r\}$-similar. We say that y apes x (boundedly) over $\{k_r\}$ if there exists a (bounded) sequence ξ wandering slowly over $\{k_r\}$ and such that $\{y_k - x_k\xi_k\}$ converges; and ξ is called an aping factor of y, relative to x. [For an extensive development of related ideas, see K. Zeller, Math. Z. **56** (1952), 134–151; MR **14**, 158.] If A is a regular T-matrix, $k_0 = n_0 = 0$, and k_1 is any positive integer, a step-by-step process yields integers n_1, k_2, n_2, k_3, n_3, $\cdots$ such that $\{k_r\}$ and $\{n_r\}$ are increasing, and

$$\lim_{r \to \infty} \max_{n_r \leq n < n_{r+1}} \left(\sum_{k < k_r} + \sum_{k > k_{r+1}} \right) |a_{n,k}| = 0.$$

The set of elements $a_{n,k}$ which occur in this expression is called the $\{k_r, n_r\}$-trim of A, and the above relation may be expressed by saying that the $\{k_r, n_r\}$-trim of A tends to 0.

The following results are then established.

Theorem 1 (Principle of Aping sequences): Let A be a regular T-matrix, and let $\{k_r, n_r\}$ be a sequence of index pairs such that the $\{k_r, n_r\}$-trim of A tends to 0. Let $x \in m$, and let $y_k = x_k\xi_k + c_k$, where ξ is bounded and wanders slowly over $\{k_r\}$, and where $c_k \to \alpha$; and let $s = Ax$, $t = Ay$. Then t has a representation $t_n = s_n\eta_n + \gamma_n$, where $\gamma_n \to \alpha$, where η wanders slowly over $\{n_r\}$, and where ξ and η are $\{k_r, n_r\}$-similar.

Theorem 2: Let x and a denote a bounded divergent sequence and a complex number, respectively, and let $\{k_r\}$ be an increasing sequence of integers such that $x(k_{2r}) \to \alpha$ and $x(k_{2r+1}) \to \beta$ $(\alpha \neq a,\ \beta \neq a,\ \alpha \neq \beta)$. Then there exists a regular T-matrix A such that (i) $Ax \to a$, (ii) Ay converges if and only if y apes $\{x_k - a\}$ over $\{k_r\}$, (iii) $\inf |a_{n,n}| > 0$ and $a_{n,k} = 0$ for $k > n$. Theorem 2′: If the hypothesis $\alpha \neq a$ is removed from Theorem 2, the theorem remains valid provided (ii) is replaced by the weaker statement (ii)′ each sequence evaluated by A apes $\{x_k - a\}$ over $\{k_r\}$, and A

evaluates each bounded sequence that apes $\{x_k - a\}$ over $\{k_r\}$.

Theorem 3: If $k_r \to \infty$, there exists a regular row-finite T-matrix which evaluates some bounded divergent sequences, and which evaluates no sequence that does not ape $\{1\}$ over $\{k_r\}$.

Theorem 4: There exists a regular T-matrix A which evaluates some bounded divergent sequences and has the following property: If $Ay \to 0$, then the set of limit points of y is either the origin, or a circular disc centered at the origin, or the entire plane.

Theorem 5: If x is a bounded divergent sequence, if the elements of $\{k_r\}$ are sufficiently far apart, and if a and b are two constants $(a \neq b)$, then no divergent sequence y $(Y \neq X)$ apes both $\{x_k - a\}$ and $\{x_k - b\}$ over $\{k_r\}$.

Theorem 6: If S is a system of mutually consistent regular T-matrices such that each bounded sequence is evaluated by some A in S, and such that to each pair of distinct points X and Y in $m/L - \{C\}$ there correspond two matrices A and B in S whose convergence fields in $m/L - \{C\}$ are disjoint and contain X and Y respectively, then the system S can be enlarged to a consistent system S^* whose convergence fields determine a Hausdorff topology without discrete points in $m/L - \{C\}$.

Theorem 7: There exists a system S of regular T-matrices whose bounded convergence fields determine a Hausdorff topology without discrete points in $m/L - \{C\}$.

Theorem 8: If A is a regular T-matrix, its convergence field in $m/L - \{C\}$ is a closed set. Corollary. The neighbourhoods in $m/L - \{C\}$ have no boundary points.

Theorem 9: If E is a neighbourhood of a point X in $m/L - \{C\}$, there exists a family $\{E_f\}$ of neighbourhoods of X (where the suffix f ranges over all real numbers in $[0, 1]$) such that $E_f \subset E_g \subset E$ whenever $0 \leqq f \leqq g \leqq 1$, and such that each E_f contains an open set which meets none of the sets E_g $(g < f)$.

Theorem 10: If E is a neighbourhood of a point X in $m/L - \{C\}$, there exists a family $\{E_f\}$ $(0 < f \leqq 1)$ of neighbourhoods of X such that each of the sets E_f contains an open set which meets none of the sets E_g $(0 < g \leqq 1, g \neq f)$.

Theorem 11: Every open set in $m/L - \{C\}$ is the union of $2^{\aleph_0}$ disjoint open sets.

Several interesting deductions from these theorems are given; see, e.g., Theorem 1, and Theorem 7 with its two proofs, the second proof resting on an unproved set-theoretic hypothesis. *R. G. Cooke* (London)

813a:

Matthews, G. Non-associative rings of infinite matrices. Nederl. Akad. Wetensch. Proc. Ser. A **60** = Indag. Math. **19** (1957), 584–589.

813b:

Matthews, G. Generalized rings of infinite matrices. Nederl. Akad. Wetensch. Proc. Ser. A **61** = Indag. Math. **20** (1958), 298–306.

813c:

Mathews, G. Bounds on generalised rings of infinite matrices. Nederl. Akad. Wetensch. Proc. Ser. A **61** = Indag. Math. **20** (1958), 354–361.

813d:

Matthews, G. A class of infinite matrices. Nederl. Akad. Wetensch. Proc. Ser. A **61** = Indag. Math. **20** (1958), 457–459.

813e:

Matthews, G. An unrestricted G-ring of infinite matrices. Nederl. Akad. Wetensch. Proc. Ser. A **61** = Indag. Math. **20** (1958), 554–556.

813f:

Matthews, G. Generalised rings of infinite matrices. II. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 45–51.

Die oben angeführten Arbeiten seien der Reihe nach mit (I)–(VI) bezeichnet.

Unter Benützung der üblichen Matrizen-Addition und -Multiplikation heißt eine Menge R unendlicher Matrizen $A, B, C, \cdots$ ein Ring, wenn gilt: (i) $A + B \in R$, $A \cdot B \in R$; (ii) $A(BC) = (AB)C$; (iii) die beim Produkt $A \cdot B$ auftretenden unendlichen Reihen konvergieren alle absolut. Wird nur (i) gefordert, so heißt R ein g-Ring; sind dabei (ii) und (iii) nicht erfüllt, so heißt R ein unrestricted g-Ring. Gelten (i) und (iii), aber nicht (ii), so heißt R ein absoluter g-Ring, und schließlich heißt R ein assoziativer g-Ring, wenn mindestens (i) und (ii) gelten.

Die Theorie der Ringe R ist durch Arbeiten von Köthe-Toeplitz [J. Reine Angew. Math. **171** (1934), 193–226] und Allen [J. London Math. Soc. **31** (1956), 374–376; MR **18**, 31] genau studiert; vgl. auch Kapitel VI in Cooke [*Linear operators*, Macmillan, London, 1953; MR **15**, 719]. Der Verfasser zeigt, daß sich ein großer Teil der Köthe-Toeplitzschen Theorie auf g-Ringe übertragen läßt.

Zunächst enthalten die Arbeiten verschiedene Beispiele. Nachdem Vermes [Indag. Math. **14** (1952), 245–252; MR **14**, 7] erstmals einen nicht-assoziativen g-Ring unendlicher Matrizen angegeben hatte, gibt der Verf. in (III, § 3) ein weiteres Konstruktionsprinzip dafür an. Darüber hinaus werden in (I, § 3), (II, § 5) und (V, § 2) sogar unrestricted g-Ringe konstruiert, während in (III, § 2) ein nicht-assoziativer g-Ring angegeben wird, in dem jede Matrix A eine 'Schranke' $|A|$ besitzt. Da jene auch als Norm in R dienen kann, entsteht eine nichtassoziative Banach-Algebra unendlicher Matrizen. Schließlich enthält (III, § 4) einen g-Ring, wo (iii) falsch ist, aber jede Matrix A eine 'Schranke' $|A|$ besitzt.

Die Untersuchung von g-Ringen ist hauptsächlich in (II) und (VI) enthalten. Ist α ein Folgenraum, und sind $x = \{x_i\}$ und $y = \{y_i\}$ $(i = 0, 1, 2, \cdots)$ komplexe Zahlenfolgen, so heiße

$$\alpha^* = \{y \mid \textstyle\sum x_i y_i \text{ ist absolut konvergent für alle } x \in \alpha\}$$

bzw.

$$\alpha^\dagger = \{y \mid \textstyle\sum x_i y_i \text{ ist konvergent für alle } x \in \alpha\}$$

der Dual-Raum bzw. g-Dual-Raum von α; gilt $\alpha^{**} = \alpha$ bzw. $\alpha^{\dagger\dagger} = \alpha$, so heißt α perfekt bzw. g-perfekt; α heißt normal, wenn $x \in \alpha$, $|y_i| \leq |x_i|$ impliziert $y \in \alpha$. Sind α, β zwei Folgenräume, so heißt die Matrix $A = (a_{ik}) \in M(\alpha, \beta)$, wenn für jedes $x \in \alpha$ alle Reihen (1) $y_i = \sum_k a_{ik} x_k$ gewöhnlich konvergieren und $y \in \beta$ ist; statt $M(\alpha, \alpha)$ schreiben wir $M(\alpha)$. Wird in (1) absolute Konvergenz verlangt, so entstehen die Teilmengen $\alpha \to \beta$ und $\Sigma(\alpha)$ von $M(\alpha, \beta)$ und $M(\alpha)$. Schließlich heißt ein g-Ring maximal, wenn er keine echte Teilmenge eines g-Rings ist. Einige typische Resultate: (a) $\alpha^{\dagger\dagger} \supseteqq \alpha$, $\alpha^{\dagger\dagger\dagger} = \alpha^\dagger$; (b) $\alpha \subseteqq \beta$ ergibt $\alpha^\dagger \supseteqq \beta^\dagger$; (c) $\alpha^* = \alpha^\dagger$ und $\alpha \to \beta = M(\alpha, \beta)$, falls α normal ist; (d) eine Bedingung dafür, daß $M(\alpha)$ ein assoziativer g-Ring ist; (e) ist α g-perfekt und $M(\alpha)$ ein g-Ring, so ist $M(\alpha)$ maximaler g-Ring. Aus (e) folgt, daß die Klasse Δ aller

konvergenzerhaltenden R-R-Matrizen ein assoziativer, maximaler g-Ring ist.—In (VI) werden vor allem die speziellen g-Ringe $M(\tau)$, $M(\lambda)$, $M(\mu)$, $M(\psi)$ studiert, wobei bezeichnet: $\tau=\{x|\sum x_i \text{ konvergent}\}$, $\lambda=\{x|\sum x_i$ hat beschränkte Teilsummen$\}$, $\psi=\{x|\sum|x_{i+1}-x_i|<\infty\}$, $\mu=\psi\cap Z$, wo $Z=\{x|x_i\to 0\}$. Die genannten g-Ringe erweisen sich als assoziative, maximale g-Ringe.

D. Gaier (Stuttgart)

814a:

Chillingworth, H. R. Generalised "dual" sequence spaces. Nederl. Akad. Wetensch. Proc. Ser. A **61** = Indag. Math. **20** (1958), 307–315.

814b:

Chillingworth, H. R. Some further results on generalised "dual" sequence spaces. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 1–10.

Die Arbeiten hängen eng mit denen von Matthews zusammen [vorstehendes Referat]. Zu gegebenem Folgenraum α wird sein dualer Raum $\alpha^\dagger$ ausführlich studiert, wobei aber der Begriff der g-Dualität statt der durch α^* gegebenen Köthe-Toeplitz-Dualität zugrunde gelegt wird. Ein Ergebnis sei angeführt: Jede in einem Folgenraum erklärte distributive und (bei geeigneter Topologie) stetige Operation wird durch eine Matrix-Transformation vermittelt. Wegen der komplexen Terminologie muß auf die Arbeiten selbst verwiesen werden.

D. Gaier (Stuttgart)

815:

Heuser, Harro. Über die Iteration Rieszscher Operatoren. Arch. Math. **9** (1958), 202–210.

A bounded linear operator A on a complex Banach space X is called a Riesz operator if for complex $\lambda\neq 0$ the following conditions are satisfied: (i) the dimension of the null space of $A-\lambda I$ is finite and equals the dimension of the quotient space X modulo the range of $A-\lambda I$; (ii) for some integer $p=p(\lambda)$ the nullspace of $(A-\lambda I)^p$ coincides with the nullspace of $(A-\lambda I)^{p+1}$, and for some integer $q=q(\lambda)$ the range of $(A-\lambda I)^q$ coincides with the range of $(A-\lambda I)^{q+1}$; (iii) the range of $A-\lambda I$ is closed.

The smallest integers p and q satisfying condition (ii) may be shown to be equal. Their common value will be called the chain length belonging to $A-\lambda I$. The Riesz operators were studied in detail in the author's dissertation [Tübingen, 1956]. From the classical results of F. Riesz it is obvious that they represent a generalization of the class of linear bounded completely continuous operators.

The main object of the present paper is to generalize to Riesz operators results concerning iteration procedures which were established by H. Wielandt for integral operators [Math. Z. **50** (1944), 93–143; MR **8**, 157]. Theorems 1 and 2 give characterizations of Riesz operators in terms of the spectrum and the nature of the resolvent, respectively. Theorems 3 and 4 lay the foundation for the iteration procedure: let A be a Riesz operator, $\mathfrak{R}_n(\lambda)$ the nullspace and $\mathfrak{L}_n(\lambda)$ the range of the $(A-\lambda I)^n$. Let λ_μ $(\mu=1, 2, \cdots)$ with $|\lambda_\mu|\geqq|\lambda_{\mu+1}|$ denote the eigenvalues of A which are different from zero, and p_μ the chain length belonging to $A-\lambda_\mu I$. Then for any integer m (for which there are at least m eigenvalues $\neq 0$) there exists a direct decomposition

$$X=\sum_{i=1}^{m}\mathfrak{R}_{p_i}(\lambda_i)+\mathfrak{R}_m$$

where $\mathfrak{R}_m$ is invariant under A, and where the spectrum of the restriction A_m of A to $\mathfrak{R}_m$ is obtained from the spectrum of A by leaving out $\lambda_1, \lambda_2, \cdots, \lambda_m$. Moreover, for any $\rho>|\lambda_{m+1}|$ the norm of the nth power of A_m is majorized by $\alpha_\rho\rho^n$ where α_ρ is a suitable positive number not depending on n.

Theorem 5 contains the basic result concerning iterations: in addition to the assumptions and definitions of theorem 4, let P_{λ_μ} be the projection defined by

$$P_{\lambda_\mu}=-\frac{1}{2\pi i}\int(A-\lambda I)^{-1}d\lambda,$$

where the integration is extended over a circle with center λ_μ whose radius a satisfies the inequalities $a<|\lambda_\mu|$, $a<|\lambda_\mu-\lambda_\nu|$ for $\mu\neq\nu$. Let x be a fixed element of X such that not $P_{\lambda_j}(x)=0$ for all j, and let μ be the smallest index such that $P_{\lambda_\mu}(x)\neq 0$. Finally let $\kappa_1=\lambda_\mu$, $\kappa_2, \cdots, \kappa_t$ be those eigenvalues for which $|\kappa_\tau|=|\lambda_\mu|$ and $P_{\kappa_\tau}(x)\neq 0$ $(\tau=1, 2, \cdots, t)$. Then for some $\sigma\geqq 0$ and some e_τ in the nullspace of $A-\kappa_\tau I$ $(\tau=1, 2, \cdots, t$; not all $e_\tau=0)$,

$$A^n(x)=n^\sigma\left\{\sum_{\tau=1}^{t}\kappa_\tau^n e_\tau+\frac{|\kappa_1|^n}{n}f_n\right\}$$

with an $f_n\in X$ whose norm has a bound independent of n. Moreover a subsequence of $A^n(x)/\|A^n(x)\|$ is constructed which converges to a normed element of the space spanned by the e_τ.

If $t=1$, this subsequence converges to a normed eigenelement of A with an eigenvalue $\kappa\neq 0$ which in turn is the limit of the sequence $\|A^n(x)\|/\|A^{n-1}(x)\|$. (Theorem 6.)

For the final theorem 7 the following assumptions are made in addition to those of theorem 5: all eigenvalues of A are real and at least one is different from zero. Assume now that x is an element of X which is not in the intersection of the ranges of $(A-\lambda_\mu)^{p_\mu}$ $(\mu=1, 2, \cdots)$. Then $A^{2n}(x)/\|A^{2n}(x)\|$ converges to an element h of X, and $\|A^{2n+2}(x)\|/\|A^{2n}(x)\|$ converges to a positive number ω. Moreover, either $h+A(h)/\sqrt{\omega}$ is an eigen-element to $\sqrt{\omega}$, or $h-A(h)/\sqrt{\omega}$ is an eigen-element to $-\sqrt{\omega}$.

E. H. Rothe (Ann Arbor, Mich.)

816:

Heuser, Harro. Über Eigenwerte und Eigenlösungen symmetrisierbarer finiter Operatoren. Arch. Math. **10** (1959), 12–20.

A bounded linear operator A on a Hilbert space X is called finite if for each complex $\lambda\neq 0$ the dimension of the null space of $A-\lambda I$ is finite and agrees with the dimension of the quotient space X modulo the range of $A-\lambda I$. (I is the identity operator on X.) A is called symmetrizable if there exists a symmetric operator H defined on X such that HA is symmetric. If in addition no eigenelement of A (belonging to an eigenvalue $\lambda\neq 0$) annihilates H then A is called completely symmetrizable. The main result of the paper is the following: let A be bounded, finite, symmetrizable by a non-negative H, and suppose $HA\neq 0$. Let x be an element of X subject only to the restriction that $HAx\neq 0$. Then for $n\to\infty$ the sequence $A^{2n}x/\|A^{2n}x\|$ converges to an element h of X, and the sequence $\|A^{2n+2}x\|/\|A^{2n}(x)\|$ to a positive number ω. At least one of the two values of $\sqrt{\omega}$ is an eigenvalue, and for such a value λ of $\sqrt{\omega}$, $h+Ah/\lambda$ is an eigenelement to λ.

The proof consists in verifying that the assumptions of theorem 7 of an earlier paper by the author [reviewed above] are satisfied. For this purpose he proves a number of theorems and lemmas showing that the operators

treated have, with respect to the eigenvalue and expansion problem, properties which are quite analogous to those well known in the case of symmetric, completely continuous operators.

Finally it is shown the main result remains true if the assumption $H \geqq 0$ is replaced by the assumption that H commutes with A. *E. H. Rothe* (Ann Arbor, Mich.)

817:

Guy, Roland. Sur une extension d'un théorème de F. Riesz et la décomposition spectrale des opérateurs Hermitiens. Rend. Mat. e Appl. (5) **17** (1958), 192–209; errata 493–494.

Proofs and applications of theorems announced in C. R. Acad. Sci. Paris **246** (1958), 2098–2101 [MR **20** #3442]. *M. Jerison* (Lafayette, Ind.)

818:

Bittner, R. On certain axiomatics for the operational calculus. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 1–9.

Let S be an endomorphism of an abelian group which is such that $Sx=y$ has a unique solution when x is subject to a condition $\gamma x=a$, γ being an endomorphism. A formalism is given for the solution of polynomial equations $\sum A_r S^r x=f$ subject to conditions $\gamma S^r x=c_r$ when the polynomials satisfy conditions of factorisation and commutability of the coefficients. These results are applied to ordinary and partial differential equations.

J. L. B. Cooper (Cardiff)

819:

de Leeuw, K. Linear spaces with a compact group of operators. Illinois J. Math. **2** (1958), 367–377.

As a generalization of harmonic analysis on compact abelian groups, the author considers a complete locally convex topological complex linear space A (called a G-space) where a homomorphism $\sigma \to T_\sigma$ of a compact abelian group G into a group of linear operators on A is defined such that $(\sigma, x) \to T_\sigma(x)$ is continuous as a mapping from $G \times A$ to A.

For a character χ of G, $T_\chi(x)=\int_G T_\sigma(x)\chi^{-1}(\sigma)d\sigma$ plays the role of "Fourier coefficient" of x at χ. The topology of convergence of Fourier coefficients is called the formal topology of A. If $\{T_\chi(x): x \in A\}$ forms a finite dimensional space for every χ, A is said to be of finite type.

The main results are the following. (i) Every G-invariant closed convex set of A is also closed by the formal topology. (ii) If A and B are two metrizable G-spaces of finite type, then every linear transformation of A to B which commutes with the operations of G is continuous if and only if it is continuous with respect to the formal topologies of A and B. (iii) In case A is a Banach space, the G-dual $A^\#$ of A is defined as the totality of elements F of the dual of A for which $\sigma \to T_\sigma^*(F)$ is continuous. Then $A^\#$ is a G-space by $\sigma \to T_\sigma^*$, and if A is of finite type, $(A^\#)^\#$ is, as G-space, canonically isomorphic to A.

I. G. Amemiya (Kingston, Ont.)

820:

de Leeuw, K. Homogeneous algebras on compact abelian groups. Trans. Amer. Math. Soc. **87** (1958), 372–386.

In *Homogeneous rings of functions* [Uspehi Mat. Nauk (N.S.) **6** (1951), no. 1 (41), 91–137; Amer. Math. Soc. Transl. no. 92 (1953); MR **13**, 139; **15**, 40], G. E. Šilov introduced a class of normed rings A satisfying the following assumptions. (i) A is a Banach algebra of continuous functions on a compact abelian group G. (ii) A contains a unit and separates points on G. Let T_s be the translation operator on A sending $f(t)$ into $f(st)$. (iii) $T_s f$ is in A whenever f is in A and s is in G. (iv) The map $s \to T_s f$ is continuous from G to A for each f in A. (v) Every maximal ideal of A stems from a point of G. In the present paper the author considers algebras satisfying (i) through (iv), but not necessarily (v). He calls them 'homogeneous Banach algebras' on G. This enlarged class of algebras contains those studied by Arens and Singer [Trans. Amer. Math. Soc. **81** (1956), 379–393; MR **17**, 1226]. Let now A be a homogeneous Banach algebra. Let S denote the semigroup of characters of G which are in A. Write Hom (S, C) for the set of all multiplicative maps of S into the complex numbers. There is a natural embedding of the maximal ideal space of A as subset $M(A)$ in Hom (S, C). The author calls a norm n on A a 'homogeneous norm' if n is equivalent to the given norm on A and if the translation operators T_s are isometric under n. Theorem: Let n be any homogeneous norm on A. Then $M(A)$ is the set of those φ in Hom (S, C) with $|(\varphi, \chi)| \leqq n(\chi)$ for all χ in S. The proof is based on a paper by the author [ibid. **83** (1956), 193–204; MR **18**, 294]. Theorem: Let S be any semi-group of characters containing the unit and generating the full character group of G. There is a natural one-one correspondence between subsets of Hom (S, C) that can be a maximal ideal space $M(A)$ for some A and real-valued functions p on S satisfying: (a) $p(\chi) \geqq 1$ for all χ in S; (b) $p(\chi_1\chi_2) \leqq p(\chi_1)p(\chi_2)$; (c) $p(\chi^2)=p(\chi)^2$. The correspondence assigns to a set X the function p_X with $p_X(\chi)=\sup_{\psi \in X}|(\psi, \chi)|$ and to each p the set $X_p=\{\varphi : |(\varphi, \chi)| \leqq p(\chi)$ all χ in $S\}$. Next, denote by $P(S)$ the set of finite linear combinations of characters in S viewed as functions on G. Call a norm n on $P(S)$ G-invariant if the translation operators are isometric under it. Theorem: There is a natural one-one correspondence between the set of all homogeneous algebras on G generated by a given semigroup S of characters and the set of all equivalence classes of G-invariant norms on $P(S)$ that satisfy $n(\chi) \geqq 1$ for all χ in S. As applications of the above, certain algebras of almost periodic functions are discussed. Let a, b be real numbers, $a \leqq 0 \leqq b$. Let $((a, b))$ denote the closed strip $\{s+it : a \leqq t \leqq b\}$ of the complex plane. Let B be a Banach algebra consisting of bounded continuous functions on $((a, b))$ which are analytic on the interior. Call B an $((a, b))$-homogeneous algebra if the following hold. Let n be the norm on B. B contains all functions $e^{i\lambda z}$, λ real and $n(e^{i\lambda z})$ is measureable in λ. If $f \in B$ and s real, then $f(s+z) \in B$ and $n(f(s+z))=n(f)$. $\{f(s+z)$: all real $s\}$ is a totally bounded subset of B for each f in B. B convergence implies pointwise convergence on $((a, b))$. Let G be the almost periodic compactification of the reals. Thus G is a compact abelian group with the reals R as dense subgroup. For each f in B put f equal to the function on G obtained by restricting f to R and extending the resulting almost periodic function to G. Let $\bar{B}$ be the algebra of continuous functions on G so obtained and define a norm $\bar{n}$ on $\bar{B}$ by $\bar{n}(\bar{f})=n(f)$. Theorem: $\bar{B}$ is a homogeneous algebra on G, and $\bar{B}$ is isomorphic and isometric to B. Theorem: Trigonometric polynomials are dense in B. Also, the author gives a number of necessary and sufficient conditions in order that $((a, b))$ be dense in the maximal ideal space of B.

J. Wermer (Providence, R.I.)

821:

Thoma, Elmar. Die unitären Darstellungen der universellen Überlagerungsgruppe der Bewegungsgruppe des R^2. Math. Ann. **134** (1958), 428–452.

Let $\mathfrak{G}$ be the locally compact group of all pairs of numbers, $\varphi \in R$ (R the real numbers) and $a \in C$ (complex numbers), with product defined by $(\varphi, a)(\varphi', a') = (\varphi+\varphi', e^{i\varphi}a'+a)$. If a is considered as defining a translation and φ a rotation, $\mathfrak{G}$ is seen to be the universal covering group of the group $\mathfrak{G}'$ of euclidean motions of R_2. The author gives a characterization of representations of the group $\mathfrak{G}$ into the space of unitary operators on a separable Hilbert space.

Let X be a set, $\mathfrak{R}$ a σ-field of subsets of X, μ a completely additive measure defined on $\mathfrak{R}$, $\mathfrak{H}_0$ a separable Hilbert space with orthonormal basis $e_1, e_2, \cdots$, and let $n(x)$ be an integral-valued (including ∞) function on X, μ-measurable with $\mu\{x : n(x) = 0\} = 0$. To each $x \in X$ assign the Hilbert space $\mathfrak{H}_0^{n(x)}$ spanned by $e_1, e_2, \cdots, e_{n(x)}$. Define $e_i(x) = \chi_{S_i} e_i$, where χ_{S_i} is the characteristic function of the set $S_i = \{x : n(x) \geq i\}$. Let $\mathfrak{M}$ be the set of all functions $f(x)$, $f(x) \in \mathfrak{H}_0^{n(x)}$ for each $x \in X$ with $(f(x), e_i(x))$ μ-measurable for every i. By using these functions, the von Neumann direct integral $\mathfrak{S} = (\int \mathfrak{H}_0^{n(x)} d\mu, \mathfrak{M})$ can be formed [Ann. of Math. (2) **50** (1949), 401–485; MR **10**, 548]. This particular type of direct integral is called a normed direct integral and is denoted by $\mathfrak{S} = (\mu, n)$. If $f \in (\mu, n)$, the projection $E(M)f = \chi_M f$, $M \in \mathfrak{R}$, is called the spectral measure belonging to (μ, n). If to each $x \in X$ a bounded linear operator $\mathfrak{A}(x)$ is defined, $\mathfrak{A}(x)$ is a bounded operator field in case $\mathfrak{A}(x)f(x) \in \mathfrak{M}$ for every $f \in \mathfrak{M}$ and $\|\mathfrak{A}(x)\| \leq B$ for some $B > 0$, μ almost everywhere. A bounded linear operator A in $\mathfrak{S}$ is called decomposable if there exists a bounded operator field $\mathfrak{A}(x)$ with $(Af)(x) = \mathfrak{A}(x)f(x)$.

Let $(U_g, \mathfrak{H})$ be a unitary representation of $\mathfrak{G}$ into a separable Hilbert space $\mathfrak{H}$ (i.e., a map of $\mathfrak{G}$ into the set of unitary operators on $\mathfrak{H}$ such that $g \to U_g x$ is continuous for each $x \in \mathfrak{H}$ and $U_{g_1 g_2} = U_{g_1} U_{g_2}$, $U_e = I$). Two such representations $(U_g, \mathfrak{H})$, $(U_g', \mathfrak{H}')$ are equivalent if there exists a unitary operator V from $\mathfrak{H}$ to $\mathfrak{H}'$ such that $VU_gV^{-1} = U_g'$ for every $g \in \mathfrak{G}$. The author shows that every unitary representation of $\mathfrak{G}$ on a separable Hilbert space is equivalent to a certain type of representation on a normed direct integral. Let $X = I \times C$ (I the closed unit interval), let

$$M_0 = \{(x, 0) : (x, 0) \in I \times C\},$$
$$M_1 = \{(x, z) : (x, z) \in I \times C,\ z \neq 0\}$$

and let

$$\mathfrak{S}_0 = \{f : f \in \mathfrak{S},\ f = \chi_{M_0} f\}, \quad \mathfrak{S}_1 = \{f : f \in \mathfrak{S},\ f = \chi_{M_1} f\}.$$

It is shown that every representation $(U_g, \mathfrak{S}_0)$ of $\mathfrak{G}$ into $\mathfrak{S}_0$ is equivalent to one of the form $[\mu, n_R]$ defined by $(U_{(\varphi,a)}f)(x) = e^{ix\varphi} f(x)$ for some n_R and measure μ, and that every representation $(U_g, \mathfrak{S}_1)$ of $\mathfrak{G}$ into $\mathfrak{S}_1$ is equivalent to one of the form $[\tilde{\mu}, n_{R^+ \times I}]$ defined by $(U'_{(\varphi,a)}f)(x, z) = e^{i\mathrm{Re}(\bar{z}a)} e^{i\varphi x} f(x, e^{-i\varphi} z)$ for some function $n_{R^+ \times I}$ and measure $\tilde{\mu}$, where one defines $n_{R^+ \times I}(x, z) = n_{R^+ \times I}(x, |z|)$ for $|z| > 0$ (R^+ is the positive reals). The author's principal theorem states that any unitary representation of $\mathfrak{G}$ into a separable Hilbert space is equivalent to one of the types $[\mu, n_R]$, $[\tilde{\mu}, n_{R^+ \times I}]$ or $[\mu, n_R] \oplus [\tilde{\mu}, n_{R^+ \times I}]$ for a suitable choice of n_R, $+\times n_{RI}$, μ, $\tilde{\mu}$. $[\mu, n_R] \oplus [\tilde{\mu}, n_{R^+ \times I}]$ is equivalent to $[\mu', n_R'] \oplus [\tilde{\mu}', n'_{R^+ \times I}]$ if and only if μ and μ' have the same null sets and $\tilde{\mu}, \tilde{\mu}'$ have the same null sets and $n_R = n_R'$, μ almost everywhere, $n_{R^+ \times I} = n'_{R^+ \times I}$ almost everywhere.

The author also characterizes unitary representations of the covering groups $\mathfrak{G}_n$ of $\mathfrak{G}'$, in which $U_{2\pi n m} = I$ for $m = 0, \pm 1, \pm 2, \cdots$ in a similar manner.

R. E. Fullerton (College Park, Md.)

822:

Foias, Ciprian. On a commutative extension of a commutative Banach algebra. Pacific J. Math. **8** (1958), 407–410.

Let A be a commutative Banach algebra with an approximate identity, and suppose that the Šilov boundary of A coincides with the space M of regular maximal ideals. Let $x \to T_x$ be the left regular representation ($T_x y = xy$ for $x, y \in A$) of A in $\mathscr{L}(A)$ (= the bounded linear transformations on A) and let $\tilde{A} = \{T_x | x \in A\}$, $\mathscr{A} = \{T \in \mathscr{L}(A) | TT_x = T_x T,\ x \in A\}$. Then $\mathscr{A}$ is the strong closure of $\tilde{A}$ in $\mathscr{L}(A)$. The author's main theorem is that if $\mathscr{M}$ is the space of maximal ideals of $\mathscr{A}$, then there is a homeomorphism $m \to \tilde{m}$ of M onto an open subset $\tilde{M}$ of $\mathscr{M}$ such that, for $x \in A$, (i) $\hat{T}_x(\tilde{m}) = \hat{x}(m)$, $m \in M$, (ii) $n \in \mathscr{M} - \tilde{M}$ implies $\hat{T}_x(n) = 0$. If A is semi-simple, so is $\mathscr{A}$, and it follows that a function f defined on M is a factor function ($f\hat{x} \in \hat{A}$ for all $x \in A$) if and only if $f = \hat{T}|M$ for some $T \in \mathscr{A}$.

When G is a locally compact abelian group and $A = L^1(G)$, the author proves that $\mathscr{A} = M^1(G)$ and observes that the preceding results, specialized to this A, are well known [cf. H. Cartan and R. Godement, Ann. Sci. École Norm. Sup. (3) **64** (1947), 79–99; MR **9**, 326; R. E. Edwards, Pacific J. Math. **5** (1955), 367–378; MR **17**, 283].

D. A. Edwards (Oxford)

823:

Kundt, Wolfgang. Bemerkung zu einem Satz über kommutative Banach-Algebren. Arch. Math. **9** (1958), 436–438.

Theorem: Let A be a complex algebra, and let $J = \{x | x \in A,\ F \subset h(x),\ F \cap \text{boundary } (h(x)) \text{ contains no nonempty perfect set}\}$, where F is a weak* closed set in the maximal ideal space Δ of A. Then J is an ideal. Proof: Let the boundary of a set E be denoted by $R(E)$. Then $F \cap R(h(xy)) \subseteq F \cap R(h(x))$, whence $y \in A$, $x \in J$ implies xy, $yx \in J$, since $h(x) \subset h(xy) \cap h(yx)$. On the other hand, $h(x) \cap h(y) \subset h(x+y)$, whence $F \cap R(h(x+y)) \subseteq F \cap [R(h(x)) \cup R(h(y))] \subseteq [F \cap R(h(x))] \cup [F \cap R(h(y))]$. The following lemma finishes the proof: If P is a space dense in itself, $P = B_1 \cup B_2$, B_i closed, then B_1 or B_2 has a nonempty perfect subset. Hence $x, y \in J$ implies $x+y \in J$, when the contrapositive of the lemma is applied to $F \cap R(h(x))$ and $F \cap R(h(y))$. Thus, in a regular semisimple commutative Banach algebra satisfying Ditkin's condition D, if I is a closed ideal, then the set of x such that $h(I) \subseteq h(x)$ and such that $h(I) \cap R(h(x))$ contains no nonempty perfect subset is an ideal lying in I.

B. R. Gelbaum (Minneapolis, Minn.)

824:

Waelbroeck, Lucien. Algèbres commutatives: éléments réguliers. Bull. Soc. Math. Belg. **9** (1957), 42–49.

The author extends to locally-convex algebras the theory of inverse elements, spectra, maximal ideals; this extension involves new phenomena (regular elements) not discernible in Banach algebras where all elements are

regular. The algebras A considered have a locally-convex, separated, quasi-complete topology; they are complex and have a unit; the product is continuous in each variable. These hypotheses are sufficient to prove important boundedness properties. Given an $a \in A$, the author studies the function $(a-s)^{-1}$ where the complex number s ranges over a suitable set T of the complex plane. Of critical importance are the notions of boundedness and holomorphy of this function and the latter has significance only at interior points of T. Whereas in a Banach algebra, inversion at a point implies bounded inversion in a neighborhood, this no longer follows in the more general setting. If $(a-s_0)^{-1}=b$ exists, $(a-s)^{-1}$ exists boundedly in a neighborhood of s_0 if and only if $(b-t)^{-1}$ exists boundedly in a neighborhood of $t=\infty$. An element b such that $(b-t)^{-1}$ exists boundedly in the neighborhood of $t=\infty$ is called regular. The spectrum of a regular element a, $\mathrm{sp}(a)$, consists of all s such that $(a-s)$ does not have a regular inverse. If a is not regular, $\mathrm{sp}(a)$ contains in addition $s=\infty$.

The author next introduces a function $h(a)$ $[0 \leq h(a) \leq \infty]$, which corresponds to the spectral radius of a: $h(a)=\max\{|s| \mid s \in \mathrm{sp}(a)\}$. The regular elements a are precisely those for which $h(a)<\infty$. The set of regular elements is denoted by A^*. It is possible to show that A^* is an algebra and that h is a multiplicative semi-norm on A^* $(h(ab) \leq h(a)\cdot h(b))$. If m is an ideal in A^*, then the set:

$$\{a \mid \inf h(a-b)=0, \quad b \in m\}$$

is also an ideal which contains m. Thus if m is maximal, it is h-closed in this sense. In this case, the semi-norm h induces a normed structure on the quotient field A^*/m which is therefore isomorphic to the complex numbers. This leads to the notion of the structure space of A^* which the author will describe in another place. Finally, an example is given of a non-commutative algebra in which two elements u and v are regular while neither $u+v$ nor $u\cdot v$ enjoys that property.

E. R. Lorch (New York, N.Y.)

825:

Balabanov, V. A. An application of the implicit function theorem to the eigenelements of non-linear operators. Soobšč. Akad. Nauk Gruzin. SSR **20** (1958), 3–8. (Russian)

A non-linear operator L is defined and is Fréchet differentiable, with Fréchet differential A_x, in a neighbourhood T of x_0 in a Hilbert space H: $Lx_0=\lambda_0 x_0$, and $\|x_0\|=1$. The stability of λ_0 and x_0 in small perturbations of L is discussed by a method which depends on assigning to an operator L^* on H an operator l^* defined by $l^*(x,\lambda)=(L^*x-\lambda x, \|x\|-1)$ on a domain of $H+R$, the direct sum of H and the real line R. The solutions of the equation $l^*(x,\lambda)=0$ are discussed by means of an extension to abstract spaces of the theorem on implicit functions, and it is proved that if either (i) λ_0 is not in the spectrum of A_{x_0} and $((A_{x_0}-\lambda_0 I)^{-1}x_0, x_0)\neq 0$, or (ii) L is a potential operator, A_{x_0} is completely continuous and λ_0 is in its spectrum, and x_0 is not orthogonal to the proper space of A_{x_0} for λ_0, then there are constants τ, τ_1 such that if $\sup_{x\in T}\{\|\Delta Lx\|+\|\Delta A_x\|\} \leq \tau$ there is a proper element $x_0+\Delta x$, having norm 1, of $L+\Delta L$ for the proper value $\lambda_0+\Delta\lambda_0$, where Δx and $\Delta\lambda$ are unique if $\|\Delta x\|+|\Delta\lambda| \leq \tau_1$.

J. L. B. Cooper (Cardiff)

826:

Èngel'son, Ya. L. Potential operators in linear topological spaces. Latvijas Valsts Univ. Zinātn. Raksti **20** (1958), no. 3, 27–45. (Russian. Latvian summary)

Let E be a locally convex linear topological space and let E^* be its conjugate space. An operator $F(x)$ on E to E^* is called a potential operator if there exists a (non-linear) functional $f(x)$ on E such that

$$\lim_{t\to 0}\frac{f(x+th)-f(x)}{t}=(F(x),h)$$

for all h in E, where (y,h) denotes the value of the linear functional y at the point h. The author's objective is to generalize some results of M. M. Vaĭnberg from Banach spaces to locally convex linear topological spaces. After defining curvilinear integrals $\int_L (F(x),dx)$ for functions F on E to E^*, taken along suitably defined regular curves L in E, the author establishes his main theorem, which asserts that for a continuous F to be a potential operator on a simply connected region of E, it is necessary and sufficient that the curvilinear integral of F be independent of the path.

D. H. Hyers (Los Angeles, Calif.)

CALCULUS OF VARIATIONS

See also 1099.

827:

Gel'man, I. V. The minimum problem for a non-linear functional. Leningrad. Gos. Ped. Inst. Uč. Zap. **166** (1958), 255–263. (Russian)

Let Ω be a bounded region in n-space such that Sobolev's embedding theorem holds for the space $W_\alpha^{(l)}(\Omega)$, l positive integer, $\alpha>1$ [*Nektorye primeneniya funkcional'nogo analiza v matematičeskoĭ fisike*, Izdat. Leningrad. Gos. Univ., Leningrad, 1950; MR **14**, 565]. The problem is to minimize $f(u)=\int_\Omega F d\Omega$ among all vector-functions $u(x)=(u^1(x),\cdots,u^N(x))$ which vanish together with all partial derivatives $u^{i,j}_{j_1\cdots j_n}(x)$, $j_1+\cdots+j_n=j$, $j=1,\cdots,l-1$, $i=1,\cdots,N$, on the boundary of Ω, where F depends on x, u, and partial derivatives $u^{i,j}_{j_1\cdots j_n}$ of all orders up to l. Let F be strictly convex in the variables u, $u^{i,j}_{j_1\cdots j_n}$ and satisfy the growth conditions:

$$K_1\sum_{i=1}^{N}\left[\sum_{l_1+\cdots+l_n=l}|u^{i,l}_{l_1\cdots l_n}|^2\right]^{\alpha/2} \leq F \leq K_2\sum_{i=1}^{N}\left[\sum_{j=0}^{l}|u^{i,j}_{j_1\cdots j_n}|^2\right]^{\alpha/2},$$

$K_1>0$, $K_2>0$. Then a unique solution in $W_\alpha^l(\Omega)$ exists. If $\alpha>2$ and there are suitable bounds on the partial derivatives of F of first and second orders, then it is shown using Gâteau differentials that every minimizing sequence tends strongly in $W_\alpha^{(l)}(\Omega)$ to the solution.

W. H. Fleming (Providence, R.I.)

828:

Bellman, Richard. Some new techniques in the dynamic programming solution of variational problems. Quart. Appl. Math. **16** (1958), 295–305.

The author considers the problem of determining the minimum or maximum of a functional $H[x_1(T),\cdots,x_k(T)]$

over all functions $y_j(t)$, $0 \leq t \leq T$, $j = 1, \cdots, N$, connected to functions $x_i(t)$, $0 \leq t \leq T$, $i = 1, \cdots, N$, $N \geq k$, by means of the linear differential equations

$$\frac{dx_i}{dt} = \sum a_{ij}(t)x_j(t) + \sum_j b_{ij}(t)y_j(t) \quad (i = 1, \cdots, N)$$

($\sum_j$ ranging over $j = 1, \cdots, N$), and satisfying constraints of the types $m_i \leq y_i(t) \leq M_i$, $0 \leq t \leq T$, $i = 1, \cdots, N$, and $\int_0^T K_j(y_1, \cdots, y_N)dt \leq b_j$, $j = 1, \cdots, l$, $l \leq N$. A technique of successive approximations is proposed in the line of the author's dynamic programming. Two points are worth mentioning. One is that if $x_i(0) = c_i$, $i = 1, \cdots, N$, and the maximum of H under the given constraints is thought of as a function $f(c_1, \cdots, c_N, T)$, then f satisfies a partial differential equation whose characteristics are given by the usual Euler equations of the problem. The other is that the method involves the computation of functions of k variables and this is of advantage when $k = 1$, 2, and n is large. *L. Cesari* (Baltimore, Md.)

829:

Malinskiĭ, K. K. The form of the second variation of a multiple integral with varied domain of integration. Vestnik Leningrad. Univ. **14** (1959), no. 1, 140–144. (Russian. English summary)

The multiple integral in question is

$$I(\alpha) = \int_{T^\alpha} F(x, u(x, \alpha), u_x(x, \alpha))dx,$$

$x = (x_1, \cdots, x_m)$, $u_x = (u_{x_1}, \cdots, u_{x_m})$, where both the real valued function $u(x, \alpha)$ and the boundary S^α of the region T^α depend smoothly on the real parameter α.

W. H. Fleming (Providence, R.I.)

830:

Grinberg, È. Ya. A geometric variational problem. Latvijas Valsts Univ. Zinātn. Raksti **20** (1958), no. 3, 153–164. (Russian. Latvian summary)

The problem is to determine the extremals for a curvilinear integral $\int f\,ds$ in n-space R_n, where f is a function of the curvatures $k_1, \cdots, k_{n-1}$ and their derivatives with respect to s of orders 1, $\cdots$, r. After the Euler-Lagrange equations are obtained, it is shown that in various particular cases the extremals can be found by quadratures from solutions of the Euler-Lagrange equations. *W. H. Fleming* (Providence, R. I.)

831:

Lombardi, Lionello. Sulla semicontinuità degli integrali di Fubini-Tonelli. Ann. Scuola Norm. Sup. Pisa (3) **12** (1958), 129–153.

Riprendendo il problema della semicontinuità dei cosidetti integrali di Fubini-Tonelli, già studiato da S. Faedo [Ann. Mat. Pura Appl. (4) **23** (1944), 69–121; Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **4** (1943), 223–249; MR **7**, 525] e dal recensore [Ann. Scuola Norm. Sup. Pisa (3) **2** (1948), 1–38; MR **12**, 267] l'A. dà una nuova interessante e assai generale condizione sufficiente. Diciamo bilinea ordinaria per l'integrale

$$\text{(I)} \quad I(y_1, y_2) = \int_a^b \int_c^d f(x, z, y_1(x), y_2(z), y_1'(x), y_2'(z))dxdz$$

ogni coppia di funzioni: $y_1(x)$, $a \leq x \leq b$, $y_2(z)$, $c \leq z \leq d$, assolutamente continue e tali da rendere finito l'integrale (I) nell'ipotesi che f sia continua insieme alle sue derivate parziali $f_{y_1'}$, $f_{y_2'}$, $f_{y_1'y_1'}$, $f_{y_2'y_2'}$, $f_{y_1'z}$, $f_{y_2'z}$ per (x, z, y_1, y_2) variabile in un dominio A dello spazio (x, z, y_1, y_2) e y_1' e y_2' qualunque. Diciamo che $I(y_1, y_2)$ è S-quasi-regolare-positivo (S-q.r.p.) se esiste una funzione $S(x, z, y_1, y_2)$ continua in A con S_{xz} e tale che

$$\begin{aligned}\mathcal{E}_S(x, z, y_1, y_2, y_1', y_2'; \bar{y}_1', \bar{y}_2') &\equiv f(x, z, y_1, y_2, \bar{y}_1', \bar{y}_2') - f(x, z, y_1, y_2, y_1', y_2') \\ &\quad - (\bar{y}_1' - y_1')f_{y_1'}(x, z, y_1, y_2, y_1', y_2') \\ &\quad - (\bar{y}_2' - y_2')f_{y_2'}(x, z, y_1, y_2, y_1', y_2') \\ &\quad - (\bar{y}_1' - y_1')(\bar{y}_2' - y_2')S(x, z, y_1, y_2) \geq 0\end{aligned}$$

per $(x, z, y_1, y_2) \in A$ e ogni y_1', y_2', $\bar{y}_1'$, $\bar{y}_2'$.

Tale $I(y_1, y_2)$ sarà poi detto anche asintoticamente bilinearizzabile (a.b.) se per ogni $R > 0$ esiste una funzione $\varphi^R(x, z, y_1, y_2, y_1', y_2')$ tale che: (1) φ^R sia continua con $\varphi^R_{y_1'}$, $\varphi^R_{y_2'}$, $\varphi^R_{y_1'y_1'}$, $\varphi^R_{y_2'y_2'}$, $\varphi^R_{y_1'z}$, $\varphi^R_{y_2'z}$; (2) risulti sempre $0 \leq \varphi^R \leq f_S$ e $\varphi^R = f_S$ per $y_1'^2 + y_2'^2 \leq R$, dove

$$f_S(x, z, y_1, y_2, y_1', y_2') = \mathcal{E}_S(x, z, y_1, y_2, 0, 0; y_1', y_2');$$

(3) esista almeno una funzione S^* e un numero $W > 0$ tali che l'integrale $\int_a^b \int_c^d \varphi^R dxdz$ sia S-q.r.p. e sia sempre $|\varphi^R_{y_1'}| \leq W(1 + |y_2'|)$; $|\varphi^R_{y_2'}| \leq W(1 + |y_1'|)$; $|S^*| \leq W$.

Il risultato principale è allora il seguente: Se esiste almeno una funzione S per cui $I(y_1, y_2)$ è S-q.r.p. e a.b., allora $I(y_1, y_2)$ è semicontinuo inferiormente su ogni bilinea ordinaria $(\bar{y}_1, \bar{y}_2)$, cioè risulta $\min \lim I(y_1, y_2) \geq I(\bar{y}_1, \bar{y}_2)$ al tendere di $y_1(x)$ a $\bar{y}_1(x)$ e di $y_2(z)$ a $\bar{y}_2(z)$ nel senso della convergenza uniforme. *E. Magenes* (Genoa)

832a:

Young, Laurent C. Existence du plan tangent en tout point intérieur O d'une surface, paramétrique ou généralisée extrême, qui rend minima une intégrale double variationnelle régulière. C. R. Acad. Sci. Paris **248** (1959), 916–919.

832b:

Young, Laurent C. Caractère local élémentaire d'une surface S, paramétrique ou généralisée extrême, qui rend minima une intégrale double variationnelle. C. R. Acad. Sci. Paris **248** (1959), 1110–1112.

In all the known existence theorems for regular double integral variational problems in parametric form, the topological type of the comparison surfaces is fixed in advance. This pair of notes outlines a promising approach to the case when comparison surfaces of all oriented topological types are admitted. It is usually immediate that a generalized surface giving the minimum exists. The main problem is to show that, near any point in the carrier of the minimizing generalized surface but not in the carrier of its boundary, the minimizing generalized surface is given by the union of a finite number of ordinary surfaces with continuously turning tangent planes. The author has told the reviewer that lemma 2.1, p. 917, is wrongly stated. A complete proof of the results announced therefore awaits a suitable replacement for this lemma.

W. H. Fleming (Providence, R. I.)

GEOMETRY

See also 690, 848.

833:

Zarovnyĭ, V. P. Interpretation of the plane axioms of affine geometry in an abstract group. Ukrain. Mat. Ž. **10** (1958), 351–364. (Russian. English summary)

Let G be a group in which a set of proper normal subgroups $\{G_\alpha\}$ is given, so that $\bigcup_\alpha G_\alpha = G$ and $G_1 \cap G_2 = \{e\}$ for every pair of different members of $\{G_\alpha\}$. Call an element of G a point and a coset of a member of $\{G_\alpha\}$ a line; then the axioms of Hilbert's group I (axioms of incidence) are satisfied. If, moreover, any two different members of $\{G_\alpha\}$ generate G, then the axiom of parallels is satisfied. Under which conditions does G satisfy the axioms of order? The author's solution of this problem is incomplete, as his corollary from theorem VII is false. Extra conditions are necessary for the validity of Pasch's axiom. The analogous problem for Desargues' theorem is also treated.

A. Heyting (Amsterdam)

834:

Marmion, A. Sur l'octuple gauche complet. Mathesis **67** (1958), 224–241.

If four pairs of points in space

$$\text{(D)} \qquad AA',\ BB',\ CC',\ DD'$$

are such that the plane determined by three points taken in any three of the four given pairs passes through a point of the fourth pair, those 8 points and the eight planes in which they lie form a complete skew octad (G. Fontené).

The 8 planes may be grouped into four pairs:

$$\text{(P)} \qquad \begin{array}{ll} P_1 = ABCD, & P_2 = ABC'D', \\ P_1' = A'B'C'D', & P_2' = A'B'CD, \\ P_3 = AB'CD', & P_4 = AB'C'D, \\ P_3' = A'BC'D, & P_4' = A'BCD'. \end{array}$$

This table shows that a point common to three planes belonging to three different pairs of (P) also lies in one of the planes of the fourth pair, that is, the complete skew octad is self-dual.

The four pairs of tetrahedrons:

$$\text{(T)} \qquad \begin{array}{ll} T_1 = A'BCD, & T_2 = AB'CD, \\ T_1' = AB'C'D', & T_2' = A'BC'D', \\ T_3 = ABC'D, & T_4 = ABCD', \\ T_3' = A'B'CD', & T_4' = A'B'C'D, \end{array}$$

into which the 8 points (D) may be grouped are Moebius couples, as is readily verified by confronting the tables (P) and (T).

The twenty-four different edges of the 8 tetrahedrons (T) are said to be the "lines of the octad".

These lines may be grouped into the following three pairs of skew quadrilaterals:

$$\text{(q)} \qquad \begin{array}{c} ABA'B',\ CDC'D';\quad ACA'C',\ BDB'D';\\ ADA'D',\ BCB'C'. \end{array}$$

Marmion considers (i) the point-net of quadrics L^2F having for base the points (D); (ii) the tangential net of quadrics L^2TF having for base the planes (P); (iii) the linear complex associated with each pair of tetrahedrons (T); (iv) the three quadrics Q_1, Q_2, Q_3, which admit as generators the eight sides of the three respective pairs of quadrilaterals (q).

Of the numerous properties brought to light by the author the following are a few examples taken rather at random.

The two secants Δ, Δ' of the four lines (D) are also the secants of the four lines P_1P_1', P_2P_2', P_3P_3', P_4P_4', and they separate harmonically the points of the four pairs (D) and the planes of the four pairs (P); they are conjugate for each of the quadrics Q_1, Q_2, Q_3.

The latter three quadrics are in pairs harmonically inscribed and circumscribed to one another.

Two quadrics harmonically inscribed and circumscribed to one another determine a biquadratic curve and a biquadratic developable such that there is an infinite number of complete skew octads inscribed in the former and circumscribed to the latter.

The surface of singularities of a quadratic complex is of the fourth order and of the fourth class. The biratio (i.e., the anharmonic ratio) of the four points of intersection of a line is equal to the biratio of the four tangent planes drawn through that line.

The treatment is largely synthetic.

N. A. Court (Norman, Okla.)

835:

Kobayashi, Katsutarô. An analogue to Clifford's chain. Math. Mag. **31** (1957/58), 133–136.

Special points of the Euclidean plane depending on n lines and a complex parameter u, $|u|=1$, in complex coordinates are defined. A theorem concerning circles and points analogous to the theorem of Clifford [Clifford, *Mathematical papers*, Macmillan, London, 1882] is proved. {The author did not notice that his theorem is identical with a limit case of Clifford's theorem for $n+1$ lines if one of these lines tends to infinity with prescribed direction.}

M. Fiedler (Prague)

836:

Horadam, A. F. Involutions associated with the Burkhardt configuration in [4]. Canad. J. Math. **11** (1959), 18–33.

In a recent paper [Quart. J. Math. Oxford Ser. (2) **8** (1957), 241–259; MR **20** #2337] the author considered a configuration in S_8, associated with a collineation group of order $81 \cdot 51840$, and an associated locus whose section by a certain S_4 was the Burkhardt quartic hypersurface with 45 nodes, invariant under a group of collineations of order 51840. In the present paper the author obtains explicit forms for the $45+270$ involutions of order two in this latter group, derives transformations in S_3 corresponding to these and investigates the figure formed by their invariant spaces.

J. A. Todd (Cambridge, England)

837:

Lagrange, René. Sur le groupe ponctuel conservant la famille des coniques du plan qui ont un élément de contact donné. Ann. Sci. École Norm. Sup. (3) **74** (1957), 197–229.

This is an elementary analytic study of the group G of all plane (birational) transformations which leave invariant the 3-dimensional linear system of conics defined either by two points I, J (chapt. I), or by a line element ($I=J$, chapt. II), in which case G is shown to be

isomorphic with the group of all similarity transformations of the Euclidean 3-space. Special attention is paid to a certain type of involutory elements of G, called the "symmetries with respect to pairs of points", and to the possibility of expressing other elements in terms of products of them. {It seems to the reviewer that a real geometrical approach would lead more directly to the results. Note for example that the given linear system can be represented birationally by the system of plane sections of a quadric if $I \neq J$, and of a cone if $I = J$; in the latter case, the isomorphism mentioned above follows immediately, by duality.} *J. L. Tits* (Brussels)

838:

d'Orgeval, B. A propos du groupe de transformations projectives qui conservent une quadrique de l'espace ordinaire. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 351–362.

Dans un travail récent [analysé ci-dessus], R. Lagrange a considéré le groupe des transformations quadratiques du plan qui changent en elle-même la famille de coniques ayant un élément de contact donné. En particulier, on peut définir une involution H_{AB} satisfaisant aux conditions désirées à l'aide de deux points arbitraires A, B; à chaque point M correspond alors le conjugué harmonique M' de M par rapport à A et B sur la conique du système passant par A, B, M. L'A. reprend ici la question, soit dans le cas de R. Lagrange, soit dans le cas plus général, où les coniques considérées ont en commun deux points distincts. À l'aide d'une projection stéréographique d'une quadrique, qui dans le cas de R. Lagrange sera un cône, les transformations considérées deviennent des homographies qui changent la quadrique en elle-même; les involutions H_{AB} deviennent des involutions biaxiales. On retrouve ainsi les résultats de R. Lagrange par une voie géométrique. *E. G. Togliatti* (Genoa)

839:

Niče, Vilko. Parametrische Sextupel Reyescher tetraedraler Strahlenkomplexe eines und zweier Haupttetraeder. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II. **13** (1958), 107–120. (Serbo-Croatian summary)

L'Autore fa alcune considerazioni relative al fascio di complessi tetraedrali associati ad un medesimo tetraedro, ed alla congruenza delle rette dello spazio ordinario che segnano le facce di due tetraedri in due quaterne formanti dato birapporto. *D. Gallarati* (Genoa)

840:

Tutte, W. T. The chords of the non-ruled quadric in $PG(3, 3)$. Canad. J. Math. **10** (1958), 481–483.

Let P denote the finite 3-dimensional projective space $PG(3, 3)$ whose 40 points have homogeneous coordinates (x, y, z, t) over the field of residues mod 3, and let Q denote the non-ruled quadric $x^2+y^2+z^2-t^2=0$ in this space. Ten points of P lie on Q, and 45 lines of P meet Q in two distinct points. [An account of the geometry of Q is given by W. L. Edge, Proc. Roy. Soc. London Ser. A **222** (1953), 262–286; MR **15**, 818.]

A graph G of order 30 is obtained from this configuration by taking the 30 points of P which are not on Q as the nodes and the 45 lines of P which meet Q in two distinct points as the edges, each edge being regarded as joining the two points of P not on Q through which it passes. The author proves that every node of G has degree 3 and that G contains no circuit with fewer than 8 edges. In an earlier paper the author proved that a graph with 30 nodes which has these properties is unique; he called it an 8-cage. [W. T. Tutte, Proc. Cambridge Philos. Soc. **43** (1947), 459–474; MR **9**, 97.] It is also shown that the automorphism groups of G and Q are isomorphic.

G. A. Dirac (Hamburg)

841:

Coxeter, H. S. M. The chords of the non-ruled quadric in $PG(3, 3)$. Canad. J. Math. **10** (1958), 484–488.

W. T. Tutte proved [see #840 above] that the graph whose nodes are the 30 points of the finite space $PG(3, 3)$ which do not lie on $x^2+y^2+z^2-t^2=0$, and whose edges are the 45 chords of this non-ruled quadric, is isomorphic to the 8-cage. (The 8-cage is the simplest cubical graph having no circuit with fewer than eight edges.) In this paper the author develops Sylvester's investigation of the fifteen "duads" and fifteen "synthemes" formed by six symbols [J. J. Sylvester, *Collected mathematical papers*, vol. 1, pp. 91–102, Univ. Press, Cambridge, 1904] and Richmond's investigation of the figure of six points in general position ("hexastigm") in projective 4-spaces, in which the duads and synthemes are used to denote the edges and transversals [H. W. Richmond, Quart. J. Math. **31** (1900), 125–160], and treats the configuration of 30 points and 45 lines in $PG(3, 3)$, considered by Tutte, from this point of view. A proof of Tutte's result is contained in the paper. *G. A. Dirac* (Hamburg)

842:

Fulton, Curtis M. Linear completeness and hyperbolic trigonometry. Proc. Amer. Math. Soc. **9** (1958), 726–728.

Let Σ be the postulate system of Hilbert with the Euclidean axiom of parallelism replaced by the characteristic postulate of hyperbolic plane geometry. The author proves the uniqueness of the relation between a segment and its angle of parallelism in a geometry based on Σ, and shows that, upon generalizing the relation $e^{-x} = \operatorname{tg} \theta/2$ for x negative, hyperbolic trigonometry follows in a remarkably simple way. *C. Longo* (Parma)

843:

Berezina, L. Ya. Some questions of analytic and differential gometry in Lobačevskiĭ space. Latvijas PSR Zinātnu Akad. Vēstis **1958**, no. 11 (136), 115–126. (Russian. Latvian summary)

Various formulas for the elementary analytic and differential geometry of the Lobačevskiĭ 3-space are established. The table of contents is self explanatory: Straight lines and planes equations. Angle between two planes, two lines, plane and line. Perpendicularity. Homogeneous normalized coordinates. Scalar product. Changes of coordinates. Exterior product of two, of three points. Line element. Tangent, normal plane, osculating plane, principal normal (to a curve). Tangent plane and normal line (to a surface). Ruled surfaces. Fundamental quadratic form of a surface. Geodesic curvature on surfaces. Lines of curvature. *J. L. Tits* (Brussels)

844:

Siegel, Carl Ludwig. Über einige Ungleichungen bei Bewegungsgruppen in der nichteuklidischen Ebene. Math. Ann. **133** (1957), 127–138.

Let Γ be the group of all real matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $|A| = 1$.

The author first considers the subgroup $\{A_k\}$ of Γ generated by m elements $A_k=\begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \in \Gamma$ $(k=1, \cdots, m)$, algebraically independent over the field Q of rational numbers (i.e. the $3m$ numbers a_k, b_k, c_k are algebraically independent over Q). He remarks that the A_k are free generators of $\{A_k\}$, that A_k contains no parabolic element, and that $\{A_k\}$ is either everywhere dense in Γ or discrete according as it contains an elliptic element or not. The latter property is an easy consequence of a theorem of J. Nielsen [Mitt. Math. Ges. Hamburg 8 (1940), part 2, 82–104; MR **2**, 213] to the effect that a non-commutative subgroup of Γ, consisting of hyperbolic elements only, is discrete. In the remaining sections, several theorems are proved which provide a "qualitative refinement" of Nielsen's theorem (and, at the same time, a simplified proof of it). We state, for instance, theorem 1, using the following notations: $A, B \in \Gamma$, $A_B=BAB^{-1}$, $[A, B]=ABA^{-1}B^{-1}$, $\sigma(A)=$trace of A. Th. 1: If A, AA_B, $[A, B]$ and $[A, A_B]$ are hyperbolic, and if $\sigma(A^2)\leqq 6$, then Min $(\sigma(AA_B), \sigma(AA_B^{-1}))<-2$. (An example shows that here, -2 cannot be replaced by a smaller number.) *J. L. Tits* (Brussels)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 1230, 1239, 1240.

845:

Trustrum, G. B. A problem on domains. J. London Math. Soc. **33** (1958), 177–181.

A set D is formed by the union of a finite number of circles of radius $\geqq 1$. The area of D is given and it is shown that the least upper bound for the length of the boundary of D is attained when D consists of unit circles at most two of which intersect.

C. J. Titus (Berkeley, Calif.)

846:

Florian, A. Zu einem Satz von P. Erdös. Elem. Math. **13** (1958), 55–58.

Let P be an arbitrary point in a triangle T and let R_1, R_2, R_3 and r_1, r_2, r_3 denote the distances from P to the vertices and the edges, respectively. An inequality of P. Erdös states that $R_1+R_2+R_3\geqq 2(r_1+r_2+r_3)$ with equality only when T is equilateral and P is the centroid. The author proves that $R_1^k+R_2^k+R_3^k\geqq 2^k(r_1^k+r_2^k+r_3^k)$ for $|k|\leqq 1$ with equality only as above. Also it is shown that $R_1^k+R_2^k+R_3^k>2(r_1^k+r_2^k+r_3^k)$ for $|k|>1$. The author also proves the analogous theorem for convex quadrilaterals by making use of a result of L. Fejes Tóth which is valid for convex n-gons (namely, $R_1+R_2+\cdots+R_n\geqq(\cos \pi/n)^{-1}(r_1+r_2+\cdots+r_n)$. The analogous question for convex n-gons remains open.

C. J. Titus (Ann Arbor, Mich.)

847:

Rogers, C. A. The packing of equal spheres. Proc. London Math. Soc. (3) **8** (1958), 609–620.

In any packing of equal spheres in Euclidean n-space, we may associate with each sphere a Dirichlet region or "Voronoi polyhedron" whose interior consists of all the points that are nearer to the center of that sphere than to the center of any other sphere. Such regions, each surrounding a sphere, fit together to fill the whole space [G. L. Dirichlet, J. Reine Angew. Math. **40** (1850), 216–219]. The density of the packing may be defined as the average of the ratio of the content of a sphere to the content of the Dirichlet region that surrounds it.

The author proves that the closest possible packing of $n+1$ spheres of radius 1 is attained when their centers are at the vertices of a regular simplex of edge 2, whose content is $2^{\frac{1}{2}n}(n+1)^{\frac{1}{2}}/n!$ [cf. R. A. Rankin, Proc. Glasgow Math. Assoc. **2** (1955), 145–146; MR **17**, 523]. Part of this content is occupied by "sectors" of the $n+1$ spheres. In the notation of Schläfli [*Gesammelte Mathematische Abhandlungen*, Bd. 2, Verlag Birkhäuser, Basel, 1953; MR **14**, 833; p. 178], this part is $2^{-n}(n+1)!J_nF_n(\alpha)$, where $J_n=\pi^{\frac{1}{2}n}/\Gamma(1+\frac{1}{2}n)$ is the content of a whole sphere, $2\alpha=\sec^{-1}n$ is the dihedral angle of the regular simplex, and the "Schläfli function" F_n is defined recursively by the formulae

$$F_0(\theta)=F_1(\theta)=1, \quad F_{n+1}(\theta)=\frac{2}{\pi}\int_0^\theta F_{n-1}(\phi)d\theta,$$

$\alpha=\frac{1}{2}\sec^{-1}n$, $\phi=\frac{1}{2}\sec^{-1}(\sec 2\theta-2)$. Thus the local density of this packing is

$$\sigma_n=2^{-3n/2}(n!)^2(n+1)^{\frac{1}{2}}J_nF_n(\alpha).$$

The same density could be maintained throughout the whole space if regular simplexes could be fitted together to make a regular honeycomb $\{3, 3, \cdots, 3, p\}$, where $p=\pi/\alpha$. Then a cell of the reciprocal honeycomb $\{p, 3, \cdots, 3, 3\}$ would serve as a Dirichlet region. Setting

$$g_{p,3,\cdots,3}=2^n/F_n(\alpha), \ (-1, k)=1-k/(n+1), \ {}_{n-1}R=1$$

in formula 8.87 of the reviewer's *Regular polytopes* [Methuen, London, 1948, Pitman, New York, 1949; MR **10**, 261; p. 161], one verifies that J_n/σ_n is the proper expression for the content of such a cell $\{p, 3, \cdots, 3\}$ of inradius 1. When $n>2$, p is fractional, the regular honeycomb can only exist in a statistical sense, and σ_n is an unattained upper bound for the density. When $n=3$, this upper bound

$$\sigma_3=3\pi(2)^{\frac{1}{2}}F_3(\tfrac{1}{2}\sec^{-1}3)=(2)^{\frac{1}{2}}(3\sec^{-1}3-\pi)=0.7797\cdots$$

is better than Rankin's $0.8271\cdots$ [Ann. of Math. (2) **48** (1947), 1062–1081; MR **9**, 226; see also the paper by Coxeter, reviewed below].

An ingenious argument, suggested by H. E. Daniels, enables the author to obtain the asymptotic formula

$$\sigma_n\sim 2^{-\frac{1}{2}n}n/e,$$

which is better than Blichfeldt's bound $2^{-\frac{1}{2}n-1}(n+2)$ [Math. Ann. **101** (1929), 605–608].

The "dual" problem, of the thinnest covering of space by equal spheres, can in fact be treated by using a polytope $\{p, 3, \cdots, 3\}$ of circumradius 1 (instead of inradius 1). Since the ratio of circumradius to inradius is $\{2n/(n+1)\}^{\frac{1}{2}}$, a lower bound for the density of a covering (attained only when $n=1$ or 2) is $\{2n/(n+1)\}^{\frac{1}{2}n}\sigma_n$. When $n=3$, this unattained lower bound $(3(3^{\frac{1}{2}})/2)\sec^{-1}3-\pi)=1.431\cdots$ is, as one would expect, slightly less than the known density $1.464\cdots$ of the thinnest lattice covering [Bambah, Proc. Nat. Inst. Sci. India **20** (1954), pp. 25–52; MR **15**, 780].

The corresponding asymptotic expression is $n/e^{3/2}$. This is better than Erdös and Rogers' 16/15 [J. London Math. Soc. **28** (1953), 287–293; MR **14**, 1066].

H. S. M. Coxeter (Toronto, Ont.)

848:

Coxeter, H. S. M. Close-packing and froth. Illinois J. Math. **2** (1958), 746–758.

The author's point of view in this rather unusual paper is best explained in his own words. 'Four equal spheres are packed as closely as possible when they all touch one another, and some further spheres can be added so as to form the beginning of a pattern apparently consisting of the inspheres of the cells of a regular honeycomb $\{p, 3, 3\}$. This beginning can be continued for spheres of a suitable size in spherical (or elliptic) space with $p=5$, and in hyperbolic space with $p=6$. The conclusion is inescapable that a compressed close-packing of equal lead shot, or a froth of equal bubbles, is trying to approximate to a Euclidean honeycomb $\{p, 3, 3\}$ in which p lies between 5 and 6. The fractional value of p means that this "honeycomb" exists only in a statistical sense.'

The author shows that, if $\{p, q, r\}$ is a Euclidean honeycomb, then

$$(1) \qquad p-4/p+2q+r-4/r = 12,$$

and putting $q=r=3$ finds the value $p=5.115$. From a formal application of Euler's theorem it is deduced that the cell $\{p, 3\}$ would have 13.56 faces. This number is compared with the average 13.70 obtained by E. B. Matzke [Bull. Torrey Botanical Club **77** (1950), 222–227] for the average number of faces of 600 central bubbles in a froth.

{As the author himself hints, the use of equation (1) is to some extent arbitrary. One could have used Schläfli's criterion $\sin(\pi/p)\sin(\pi/r)=\cos(\pi/q)$, which would have led to a rather smaller value, 13.398. Nevertheless, this is a most interesting paper.}

J. A. Todd (Cambridge, England)

849:

Zaguskin, V. L. Circumscribed and inscribed ellipsoids of extremal volume. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 89–93. (Russian)

Generalizations of the theorems by Behrend [Math. Ann. **113** (1937), 713–747; **115** (1938), 397–411] on the uniqueness of ellipsoids of extremal volumes in affine space are proved here independently of the same results by Danzer, Laugwitz and Lenz [Arch. Math. **8** (1957), 214–219; MR **20** #1283]. *V. Linis* (Ottawa, Ont.)

850:

Toponogov, V. A. Evaluation of the length of a closed geodesic on a convex surface. Dokl. Akad. Nauk SSSR **124** (1959), 282–284. (Russian)

Consider a closed, non-intersecting geodesic in a two-dimensional continuously twice differentiable convex surface R_k of curvature not exceeding the positive number k. The following theorems are proved. (1) The length of the geodesic does not exceed $2\pi/\sqrt{k}$. (2) If there exists a simple closed geodesic of length $2\pi/\sqrt{k}$ on R_k then R_k is the two-dimensional sphere of radius $1/\sqrt{k}$.

For a geodesic loop on R_k the author states the following extensions. (3) The length of a geodesic loop L in R_k does not exceed $2\pi/\sqrt{k}$; the length L equals $2\pi/\sqrt{k}$ if and only if R_k is the sphere.

The author states that the analogous theorem can also be proved for doubly-connected two-dimensional Rieman manifold R_k of curvature not more than k ($k>0$). In this case the estimate of the length must be halved.

P. C. Hammer (Madison, Wis.)

GENERAL TOPOLOGY, POINT SET THEORY

851:

Sierpiński, W. Sur un problème de H. Steinhaus concernant les ensembles de points sur le plan. Fund. Math. **46** (1959), 191–194.

Steinhaus a posé le problème suivant: Existe-t-il sur le plan deux ensembles A et B, sauf le cas où un de ces ensembles est le plan tout entier et l'autre est formé d'un seul point, tels que tout ensemble situé sur ce plan et superposable avec A ait exactement un point commun avec tout ensemble situé sur le même plan et superposable avec B? L'auteur donne une réponse affirmative à ce problème à l'aide du lemme suivant: E étant un ensemble de points de puissance $<2^{\aleph_0}$ il existe un point g tel que toutes les distances de g aux points de E sont positives, distinctes deux à deux et distinctes de toutes les distances entre deux points quelconques de E.

A. J. Lohwater (Ann Arbor, Mich.

852:

Lohwater, A. J. The Moore-Young theorem in point-set topology. Arkhimedes **1956**, 22–24. (Finnish)

A plane point-set consisting of three simple arcs issuing from one point and not otherwise intersecting is called a plane triod. The present article gives an elementary proof of the theorem of R. L. Moore that only a countable number of non-intersecting triods can exist in the plane [Proc. Nat. Acad. Sci. U.S.A. **14** (1928), 85–88]. The proof is very simple and depends on the following lemma, which is a special case of a theorem of Z. Janiszewski [Prace Mat.-Phys. **26** (1913), 1–63], but is proved here, in contrast to the general case, in a completely elementary manner from a well-known theorem on Jordan curves: two triods, each of which lies in a given circle C and joins three non-intersecting arcs of the circumference of the circle (i.e. has its endpoints lying on these arcs but otherwise lies inside C), have at least one common point.

A formulation is also derived for the theorem of G. S. Young generalizing the Moore theorem to the n-dimensional case, where the n-dimensional analog of a triod is understood to be a point-set of n-dimensional space consisting of an $(n-1)$ dimensional cell and a simple arc issuing from an interior point of the cell but not otherwise intersecting it [Bull. Amer. Math. Soc. **50** (1944), 714; MR **6**, 96]. It is shown that Young's proof is based on the Alexander duality theorem and on a theorem of Whyburn, and the remark is made that it would be desirable to find an elementary proof of this theorem, at least for the case $n=3$.

M. F. Bokštein (RŽMat **1957** #6917)

853:

Pasynkov, B. Polyhedral spectra and the dimensionalities of bicompacts, especially of bicompact groups. Dokl. Akad. Nauk SSSR **121** (1958), 45–48. (Russian)

Every bicompact space is a limit of a spectral development of polyhedral spaces; if it is 0-dimensional the spectral mappings can be chosen to be onto and simplicial on subdivisions. If the bicompact space has a development into n-dimensional polyhedra ($n=0$ or 1) the dimension defined by coverings and the two inductive dimensions are equal. For arbitrary n, $n \geqq \mathrm{Ind}$ is still true.

H. Freudenthal (Utrecht)

854:

Sasao, Seiya. On the P-extension of topology. J. Math. Soc. Japan **10** (1958), 304–306.

The author obtains a sufficient condition for the formation of weak (= fine!) topologies in the sense of D. E. Cohen [Proc. London Math. Soc. **7** (1951), 219–248; MR **19**, 441] to commute with the formation of finite cartesian products, generalizing Cohen's theorem 3.2 on the product of k-spaces. *E. Michael* (Seattle, Wash.)

855:

Mrówka, S. On the topology of the spaces of closed subsets. Prace Mat. **2** (1958), 309–318. (Polish. Russian and English summaries)

With the usual definitions of the topological upper and lower limits of a sequence of subsets of a topological space X ($x \in X$ is in limsupt A_n [liminft A_n] if and only if every neighborhood of x meets infinitely many [all but finitely many] sets of $\{A_n\}$), and of convergence of such a sequence, the author expresses the characteristic functions of limsupt A_n [liminft A_n] in terms of the characteristic functions of A_n. He applies this result to prove the known theorem (theorem 1 of the paper) that any sequence of subsets of a separable space contains a convergent subsequence. Theorem 2 of the paper is a direct consequence of lemma 2.3.1 as stated without proof by E. Michael [Trans. Amer. Math. Soc. **71** (1951), 152–182; MR **13**, 54]. This theorem states that Vietoris (finite in the E. Michael terminology) topology on the space 2^X of closed subsets of X is the same regardless of the choice of the base in X used in defining that topology in 2^X. Theorem 3 proves that if X is compact metric, then the Hausdorff and Vietoris topologies on 2^X coincide. The statement of this fact can be found on p. 154 of the E. Michael paper quoted above. Theorem 4 shows that if X is compact regular, then limsupt A_n = liminft A_n if and only if the sequence $\{A_n\}$ converges in 2^X with Vietoris topology. Theorems 5 and 6 give the relations induced by continuous mappings between the spaces X, 2^X and certain cartesian products of spaces consisting of two points. *C. Masaitis* (Havre de Grace, Md.)

856:

Infantozzi, Carlos A. On a necessary condition for the generalized covering property. Bol. Fac. Ingen. Agrimens. Montevideo **6** (1957/58), 259–268. Also published as Fac. Ingen. Montevideo. Publ. Inst. Mat. Estadist. **3** (1958), 69–79. (Spanish. English summary)

The paper establishes relations among the compactness conditions studied in the author's earlier paper [same Bol. **4** (1951), 409–419; MR **14**, 571] and others, of which a specimen follows. Axiom QPC[α, β]: If A is a subset of the space in question, the cardinal number N of A lies in the interval $[\alpha, \beta]$ and $\gamma < N$ is given, then there is a point whose every neighborhood meets A in a set of cardinal number exceeding γ. *R. Arens* (Los Angeles, Calif.)

857:

Duda, R. Sur le prolongement des homéomorphies. Fund. Math. **46** (1959), 175–186; addendum, 360.

The first paper gives some results concerning the extension of homeomorphisms. The primary result of this paper is contained in theorem three. In the second note the author points out that he now recognizes theorem three as a special case of an earlier theorem of L. Zippin [Amer. J. Math. **57** (1935), 327–341]. *Wayman Strother* (Palo Alto, Calif.)

858:

Borsuk, K.; and Molski, R. On a class of continuous mappings. Fund. Math. **45** (1957), 84–98.

Let f be a continuous mapping defined on a topological space X. Then f is termed simple if for each y in $f(X)$ the set $f^{-1}(y)$ contains at most two points, and the union of all sets $f^{-1}(y)$ which contain two points is called the seam of f. The index of f is the smallest number of sets which may constitute a system of closed sets X^* covering X such that for each y in $f(X)$ and each X^* the set $X^* \cap f^{-1}(y)$ contains at most one point. And f is termed elementary if X is metric and there is a positive number ε such that any pair of distinct points in X which have the same image have a distance not less than ε. This is a study of simple mappings. Eleven examples are given. If X is a compactum and f has finite index n it is shown that the dimensions of X and $f(X)$ are equal, and f is a finite superposition of simple mappings with indices not exceeding n. If f is simple elementary, X is a locally contractible compactum of finite dimension and f has a locally contractible seam, then $f(X)$ is shown to be an absolute neighborhood retract. If f is simple, X is compact, and A is a retract of X containing the seam of f, then $f(A)$ is a retract of $f(X)$. If f is elementary simple, X is compact, the $(n-1)$st Čech homology group of the seam of f is trivial and the nth homology group of either X or its seam is trivial, it is established that the nth homology groups of X and $f(X)$ are isomorphic. *P. V. Reichelderfer* (Columbus, Ohio)

859:

Jaworowski, J. W. On simple regular mappings. Fund. Math. **45** (1958), 119–129.

A simple mapping f of a topological space X with seam X_0 [see review above] is termed regular provided f transforms sets open relative to the closure of X_0 onto open sets. If f is a simple mapping of a compact metric space it is noted that f is elementary [see review above] if and only if f is regular and its seam is closed. If f is a simple regular mapping of a compact space X with seam X_0 there is defined a space D, termed the doubling of X by f; D is shown to be the union of two subsets, each homeomorphic with X and intersecting in a set homeomorphic with the closure of X_0; and a simple interior mapping is naturally defined on D which maps D onto a set homeomorphic with $f(X)$. This doubling process is used to establish some properties of simple regular mappings of compact space. For example, it is proved that if f is a simple regular mapping of a compact space X, then X and $f(X)$ have the same dimension. And if f is a simple elementary mapping of a compact space X with seam X_0, then $f(X)$ is an absolute neighborhood retract if both X and X_0 are. *P. V. Reichelderfer* (Columbus, Ohio)

860:

Knaster, B.; et Lelek, A. Coutures et tapis. Fund. Math. **45** (1958), 186–199.

A tapis on a spherical surface S_2 is defined to be a locally connected continuum of dimension one whose complement is the union of a sequence of domains having

disjoint simple closed curves for boundaries. It is shown that every point of a tapis is a point of ramification of order $2^{\aleph_0}$. A set is defined in a square and shown to be homeomorphic to a tapis and the continuous image of a linear segment under a simple mapping [see review second above]. If Γ is a continuum on S_2 which is either S_2 itself or has a complement composed of a finite number of domains whose boundaries are disjoint simple closed curves and T is a tapis, then it is proved that there exists a mapping T onto Γ which is simple. Thus, Γ may be obtained from a linear segment by a continuous transformation which is the superposition of two simple mappings. *P. V. Reichelderfer* (Columbus, Ohio)

861:

Seki, Takejiro. An elementary proof of Brouwer's fixed point theorem. Tôhoku Math. J. (2) **9** (1957), 105–109.

The author proves Brouwer's fixed-point theorem in the form: If f maps continuously an n dimensional sphere $\|X\| \leq 1$ into itself, there exists a point X such that $f(X) = X$. The interesting proof is given without appeal to the theory of simplicial mappings but makes essential use only of a few well-known properties of real analytic functions. *C. J. Titus* (Berkeley, Calif.)

862:

Nagami, Keiô. Finite-to-one closed mappings and dimension. I. Proc. Japan Acad. **34** (1958), 503–506.

The author first proves the following theorem. Let f be a continuous mapping of a 0-dimensional metric space R onto a metric space S. If f is closed and $f^{-1}(y)$ has exactly $k\,(<\infty)$ points for each $y \in S$, then $\dim S = 0$. He then gives a number of dimension-theoretic consequences, including some known theorems of Katětov [Czechoslovak Math. J. **2** (**77**) (1952), 333–368; MR **15**, 815] and Morita [Math. Ann. **128** (1954), 350–362; MR **16**, 501], and some extensions of them. Finally, the hypotheses on f in the first theorem can be replaced by: f is open and $f^{-1}(y)$ is finite for each $y \in S$. Most of the proofs are omitted; details are to appear elsewhere. {Theorems 8 and 9 appear to be misprinted, as the set A_0 plays no part in the conclusions.} *A. H. Stone* (Manchester)

863:

Vopěnka, Petr. On the dimension of compact spaces. Czechoslovak Math. J. **8** (**83**) (1958), 319–327. (Russian. English summary)

Let dim, ind and Ind denote the covering dimension and the inductive dimensions based on neighborhoods of points and closed sets, respectively. For any $1 \leq m \leq n \leq \infty$ there exist compact spaces X and Y with $\dim X = m$, $\operatorname{ind} X = n$, $\dim Y = m$, $\operatorname{Ind} Y = n$. Previously it was known only that there are compact spaces X with $\dim X = 1$ and $\operatorname{ind} X = \operatorname{Ind} X = 2$ [A. Lunc, Dokl. Akad. Nauk SSSR **66** (1949), 801–803; MR **11**, 46; O. V. Lokucievskiĭ, ibid. **67** (1949), 217–219; MR **11**, 46]. The above result follows from this interesting proposition: Given any compact space X, $\dim X > 0$, there is a compact space $T \supset X$ such that (i) $\dim T = \dim X$, $\operatorname{Ind} T \leq 2 \operatorname{Ind} X + 1$, and (ii) there is a $t \in T$ and open set U about t such that the frontier of each open G, $t \in G \subset U$, contains a homeomorph of X. *S. Mardešić* (Zagreb)

ALGEBRAIC TOPOLOGY

See also 653, 732, 807, 840, 902, 904.

864:

Nakaoka, Minoru. Cohomology of symmetric products. J. Inst. Polytech. Osaka City Univ. Ser. A **8** (1957), 121–145.

Let X be a finite simplicial complex with a base point $*$. Let P_n denote the n-fold Cartesian product of X and $\bar{P}_n$ the reduced product (obtained from P_n by identifying to a point the set $P_n' \subset P_n$ of all $(x_1, \cdots, x_n)$ with $x_i = *$ for some i). The symmetric group $\mathfrak{S}_n$ acts on P_n and $\bar{P}_n$ in the obvious way. The orbit spaces are the symmetric product SP_n and the reduced symmetric product $\overline{SP}_n$, respectively. By means of certain explicit maps and homomorphisms the author obtains this direct sum decomposition

$$H^q(SP_n; G) \approx \sum_{m=1}^{n} H^q(\overline{SP}_m; G)$$

and hereafter studies $\overline{SP}_m$.

Let $F_n \subset P_n$ be the union of the fixed point sets for various transformations $\alpha \in \mathfrak{S}_n$. The definition of the corresponding sets $\bar{F}_n$ and $\overline{SF}_n$ is clear. $\mathfrak{S}_n$ acts on $\bar{P}_n - \bar{F}_n$ without fixed points and $\overline{SP}_n - \overline{SF}_n$ is the corresponding orbit space. Hence, one obtains a Cartan-Leray spectral sequence for finite regular coverings [see Séminaire H. Cartan, École Norm. Sup. 1950/51, 2e éd., 1955; MR **17**, 1117; Exposés 11 and 12].

Now, it is assumed that $H_q(X; Z) = 0$, for $q < r$, $r \geq 2$. By direct considerations $H_q(\bar{F}_n)$ is evaluated; this yields information about $H_q(\bar{P}_n, \bar{F}_n)$ and (by a spectral sequence argument) about $H_q(\overline{SP}_n, \overline{SF}_n)$. This enables the author to prove that $H^q(\overline{SP}_n) = 0$ and $H^q(SP_n) \approx H^q(X)$, for $0 < q \leq r+1$, $n \geq 2$ (actually, a more complicated theorem is proved, this being its corollary).

Finally, the techniques and results developed are employed to recalculate certain cohomology groups of SP_2 and SP_3 for r-spheres [M. Nakaoka, J. Inst. Polytech. Osaka City Univ. Ser. A **7** (1956), 51–102; MR **19**, 972]. *S. Mardešić* (Zagreb)

865:

Rohlin, V. A.; and Švarc, A. S. The combinatorial invariance of Pontrjagin classes. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 490–493. (Russian)

The authors prove: If the smooth manifolds M_0^n, M_1^n have isomorphic C^1-triangulations, then the induced simplicial map φ preserves the Pontryagin classes. [This has also been proved by R. Thom, see #866 below.] For the proof one may assume n odd (if necessary, multiply by S_1). There exists an integer m such that every homology class $m \cdot u'$, $u' \in H_{4k}(M_0^n)$, can be represented by a submanifold N_0^{4k} with trivial normal bundle (the dual cohomology class comes from a map into a sphere; N_0^{4k} is the pre-image of a general point). For the classes $s_{4k} \in H^{4k}(M_0^n)$ (defined by the Pontryagin classes and the Hirzebruch polynomial L_k, multiplied by its denominator α_k) one gets the relation

$$(*) \qquad (s_{4k}(M_0^n), u) = \alpha_k \cdot \tau(N_0^{4k})$$

(with τ = index). Using the map φ, one constructs a similar manifold N_1^{4k} in M_1^n. It is then shown that N_0^{4k}

and N_1^{4k} belong to the same (combinatorial) cobordism class; their indices are equal, and the theorem follows. It is stated that the result holds for open and for bounded manifolds.

Essentially the same construction yields a definition of Pontryagin classes for combinatorial manifolds: The right side of (*) is independent of the choices made, and defines s_{4k} as rational cohomology class. It follows that a certain manifold constructed by Milnor [Ann. of Math. (2) **64** (1956), 399–405; MR **18**, 498], suitably triangulated, is not combinatorially equivalent to a smooth manifold, and not even intrinsically homologous (= cobording) to one.

H. Samelson (Ann Arbor, Mich.)

866:

★**Thom, R. Les classes caractéristiques de Pontrjagin des variétés triangulées.** Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 54–67. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv + 334 pp.

The author gives a definition of Pontryagin classes (with rational coefficients) for combinatorial manifolds invariant under simplicial isomorphism and J-equivalence (two manifolds are J-equivalent if they are cobounding, in the author's sense, in such a way that they are deformation retracts of the manifold that they together bound). [Substantially the same result has been given by V. A. Rohlin and A. S. Švarc, see #865 above.] First, "piecewise differentiable spaces" are introduced; they are made up of cells that are open sets in Euclidean spaces R^k, with the cells intersecting differentiably and in general position. Every polyhedron is such a space. Differentiability concepts carry over to these spaces: differentiable map into R^k, regular point, Sard's theorem, t-regularity on a submanifold, the fibering of the inverse image, under a proper map, of a small open set of regular points, etc. A submanifold of codimension q has normal orthogonal structure if it is definable by a map into the Thom complex $M(SO(q))$. Normal Stiefel-Whitney and Pontryagin classes are then defined although there is no normal bundle in the usual sense. Pontryagin classes are now introduced axiomatically: (1) $p_0 = 1$; (2) if W is a submanifold of V with normal orthogonal structure, and injection map i, then

$$i^*(1+p_1+\cdots) = (1+n_1+\cdots)\cup(1+q_1+\cdots);$$

here the n_i are the normal classes, and the q_i are the classes of W; this is the "duality" axiom; (3) for a $4j$-manifold one has $\tau(V) = L_j(p_1, \cdots, p_j)$ with τ = index, and L_j the Hirzebruch polynomial. For the existence proof one notes that axiom (3) defines p_j for $4j$-manifolds, if the p_i, $i<j$, are known. For a general V one defines p_j on every submanifold X_{4j} with normal orthogonal structure by (2) and (3) and verifies that $p_j(X_{4j})$ depends only on the L-class of X_{4j}, and so defines a rational character of the group L_{4j} of L-classes. Finally it is shown that this character vanishes on the kernel Y^{4j} of the natural map of L_{4j} onto H_{4j} and so defines a cohomology class. The last step depends on the fact that V can be assumed of high dimension (by multiplying by a sphere) and that then the cohomology class dual to X_{4j} (multiplied by some integer) is spherical. Y^{4j} is generated by L-classes of type zero which are generated by maps of V into the Thom complex that factor through a sphere of suitable dimension.

It follows that a piecewise differentiable structure (e.g. a triangulation) can derive from a global one only if the Pontryagin classes are integral. Some remarks are made about the connection with the Hauptvermutung. In particular, Milnor's examples show that the common conjecture "Every triangulable manifold can be made differentiable" is in contradiction with the Hauptvermutung.

Some indications are given of how to extend all this to fiber maps whose fiber has the homology of Euclidean space.

H. Samelson (Ann Arbor, Mich.)

867:

Takizawa, Seizi. Some remarks on invariant forms of a sphere bundle with connexion. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29** (1955), 193–198.

Let M be a differentiable manifold and $B^{n-1}(M, S^{n-1}, O_n^+)$ be an $(n-1)$-sphere bundle over it. We denote by $B^q(M, Y^q, O_n^+)$ the associated bundle with Stiefel manifold $Y^q = O_n^+/O_q^+$ as fibre. The principal bundle associated to these is a principal bundle over M with group and fibre O_n^+, but it may also be regarded as a principal bundle over $B^q(M, Y^q, O_n^+)$ with group and fibre O_q^+. As such any connection in it regarded as a principal bundle over M induces a connection in it as a principal bundle over B^q. Using this, the author presents simpler versions of the formulas of two earlier papers [same Mem. **28** (1953), 1–10, 11–14; MR **15**, 646, 647] in which he gives differential forms in the principal bundle over M which correspond to the Whitney characteristic classes. These results are related to the Gauss-Bonnet formula, but unlike that case the forms obtained are not in the base space M.

W. Boothby (Evanston, Ill.)

868:

Allendoerfer, Carl B.; and Eells, James, Jr. On the cohomology of smooth manifolds. Comment. Math. Helv. **32** (1958), 165–179.

According to de Rham's theorem the cohomology algebra $H^*(X, R)$ of a differentiable manifold X with the real numbers R as coefficients is isomorphic to the algebra of differentiable, closed, exterior differential forms on X modulo those which are exact. By using a suitably defined module of forms with singularities, the authors have extended the de Rham theory to an isomorphism with $H^*(X, A)$ where A is any integral domain in R, e.g. $A = Z$, the integers. The details are briefly the following: Let $e(\omega)$ denote the set of singular points of a form ω whose singularities are at most a nowhere dense set on X; if ω is an $(r-1)$-form, θ an extension of $d\omega$ such that $e(\theta)$ is a closed subset of $e(\omega)$, then (θ, ω) is called a pair on X. A smooth singular chain c is admissible for (θ, ω) if $|c| \cap e(\theta) = \emptyset = |\partial c| \cap e(\omega)$, $|c|$ being the carrier of c. For an admissible chain c one defines the residue of (θ, ω): $R[(\theta, \omega), c] = \int_c \theta - \int_{\partial c} \omega$; (θ, ω) is an (A, r) pair if this residue is in A. Two such pairs are equivalent if their residues are the same on all admissible r-chains c. The equivalence classes $[\theta, \omega]$ of (A, r) pairs form a module over A. The operator d is defined by $d[\theta, \omega] = [0, \theta]$, and after a suitable definition of multiplication for such pairs one obtains a corresponding cohomology ring $H^*(X, A)$. The principal theorem is that this cohomology ring is isomorphic to the usual one of singular cohomology with coefficients in A and cup product as multiplication. The method of proof is similar to the sheaf theory of proof of

de Rham's theorem as found, for example, in F. Hirzebruch, *Neue topologische Methoden in der algebraische Geometrie* [Springer, Berlin–Göttingen–Heidelberg, 1956; MR **18**, 509].

Further remarks show how this theory may be extended to coefficients in Z_m (integers modulo m) and to open manifolds. Several applications and modifications of the main theorem are given, including a proof of the Poincaré duality and a study of the relation of this ring and the intersection ring of a manifold. In case the manifold is Riemannian it is shown that in each cohomology class of $H^*(X, A)$ is a unique (A, r) pair (θ, ω) with θ harmonic.

W. M. Boothby (Evanston, Ill.)

869:

Wu, Tsen-teh. On the mod 2 imbedding classes of a triangulable compact manifold. Sci. Record (N.S.) **2** (1958), 435–438.

In a paper of Wu Wen-tsün [Acta Math. **7** (1957), 79–101; MR **20** #3536], it was proved that if M^n is a triangulable compact differentiable manifold of dimension n with $\rho_2\Phi^N(M^n)=0$, then (1) $\overline{W}^k(M^n)=0$, $k\geqq N-n$; and that if M^n is a triangulable compact differentiable manifold of dimension n with $\overline{W}^k(M^n)=0$, $k\geqq N-n$, then (2) $Sm^kH^r(M^n; I_2)=0$, $2k+r\geqq N$.

There arise thus naturally two problems as follows. (i) Does there exist an n-dimensional compact differentiable manifold for which (2) holds but not (1)? (ii) Does there exist an n-dimensional compact differentiable manifold for which (1) holds but $\rho_2\Phi^N(M^n)\neq 0$? In this paper, the author answers these questions negatively.

Sze-tsen Hu (Palo Alto, Calif.)

870:

Matuzevichus, A. Cross-sections of twofold fiber spaces. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 272–275. (Russian)

Let $P_1: C_1\to P_1\to B$ and $P_2: C'\to P_2\to P_1$ be two Serre fiberspaces; assume C_1 and C' to be r-, resp. $(r-1)$-simple, and B 1-connected. Suppose given two sections S_1, S_2 of P_1 over the r-skeleton B^r of B, coinciding on B^{r-1}. Denote by Q_1, Q_2 the fiberspaces over B^r induced from P_2 by S_1, S_2; note that they coincide over B^{r-1}. Suppose ψ is a section of Q_1 (and Q_2) over B^{r-1}. Let $Z_1{}^r$, $Z_2{}^r$ be the first obstructions to extending ψ over B^r in Q_1, Q_2; they lie in $H^r(B^r, \pi_{r-1}(C'))$. Let D^r be the difference cocycle of S_1 and S_2, an element of $H^r(B^r, \pi_r(C_1))$. From the fiberspace over C_1, induced from P_2 by the inclusion $C_1\subset P_1$, one gets the boundary map $\Delta: \pi_r(C_1)\to\pi_{r-1}(C')$ of the homotopy sequence, and the induced map

$$\hat{\Delta}: H^r(B^r, \pi_r(C_1))\to H^r(B^r, \pi_{r-1}(C')).$$

The result of the note consists in the formula $Z_1{}^r - Z_2{}^r = \hat{\Delta}D^r$. The proof rests on the following interpretation of Δ: for $\alpha\in\pi_r(C_1)$ consider the fiberspace over the sphere S^r induced from P_2 by α and $C_1\subset P_1$; then $\Delta(\alpha)$ is the value of the obstruction on the basic cycle of S^r. {It seems to the reviewer that these two quantities differ in sign [cf. Steenrod, *Topology of fibre bundles*, Princeton Univ. Press, Princeton, N.J., 1951; MR **12**, 522; p. 180].}

Applications to a principal bundle over B, with structure group G and subgroups $\Gamma_1\supset\Gamma_2$. The spaces P_1 and P_2 are now E/Γ_1 and E/Γ_2; the fibers are $C_1=G/\Gamma_2$, $C'=\Gamma_1/\Gamma_2$.

H. Samelson (Ann Arbor, Mich.)

871:

Švarc, A. S. Geodesic arcs on Riemann manifolds. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 181–184. (Russian)

Extending Morse's classical result, Serre [Ann. of Math. (2) **54** (1951), 425–505; MR **13**, 574] proved that there are an infinite number of geodesics joining any two points of a compact, connected, simply-connected Riemannian manifold. Morse showed that the lengths of these increase without bound. The author proves that there is a subsequence of this collection of geodesics whose lengths increase no faster than an arithmetical progression. The method combines the Serre-Borel computations for the cohomology of the loop space [loc. cit. and A. Borel, ibid. **57** (1953), 115–207; MR **14**, 490] with Lusternik's application of cohomology to the calculus of variations [Trudy Mat. Inst. Steklov. **19** (1947); MR **9**, 596]. A lemma of purely topological interest is: In the space of loops based on a fixed point of the manifold, there is a homology class x and a cohomology class ξ such that $\langle x^k, \xi^k\rangle = k!$ for $k=1, 2, \cdots$. (x^k denotes the Pontryagin product of x with itself k times.)

L. W. Green (Minneapolis, Minn.)

872:

Griffiths, H. B. Local topological invariants. II. Trans. Amer. Math. Soc. **89** (1958), 201–244.

[For part I, see Proc. London Math. Soc. (3) **3** (1953), 350–367; MR **15**, 457.] This paper is organized around the presentation of the relations among various invariants of local character suggested by the homology and homotopy groups. These local invariants are defined using inverse and direct limit groups. There are also results of intrinsic interest about inverse and direct limit groups, and theorems stating that under certain conditions singular homology groups and homotopy groups will be finitely generated.

An abridged summary follows; there are four main lines of development. (1) A concept of stability of the representation of a group as an inverse (or direct) limit is introduced, and some algebraic properties of this concept are proved. (2) The theorems about the finite generation of topological invariants are: (a) If X is locally compact metric, F a compact subset, G a neighborhood of F, the image of the inclusion $H_q(F)\to H_q(G)$, $0\leq q\leq n$, of singular homology groups is finitely generated provided that X is lc_v^n and lc_s^n (i.e. locally connected in dimensions $\leq n$ in the sense of Vietoris homology and in the sense of singular homology); (b) if Y is locally compact metric, U a compact subset, then the image of the inclusion map $\pi_n(U)\to\pi_n(Y)$, $n>1$, is finitely generated provided that Y is lc_v^n and lc_s^n, and there exists a set X such that $U\subseteq$ Interior (X) and there is a chain of subsets $X=A_0\subseteq A_1\subseteq\cdots\subseteq A_n=Y$ such that the inclusion $\pi_r(A_r)\to\pi_r(A_{r+1})$ has image zero, $r=0, 1, \cdots, n-1$; (c) a similar theorem is proved for the fundamental group.

(3) If X is a locally compact metric space, x a point in X, definitions of invariants $\pi_n(x)$ of local character suggested by the homotopy groups are given using inverse limit groups, and homotopy local connectedness is defined in terms of the stability of these groups. It is shown that (a) LC^n (i.e. homotopy local connectedness in dimensions $\leq n$) at x implies lc_s^n at x; (b) a converse of (a) can be obtained if one assumes that the space is LC^1 at x; (c) a local analogue of the Hurewicz theorem is proved, that is, if the local homotopy groups $\pi_q(x)=0$, and are stable,

$q<n$, then $\pi_n(x) \approx H_n(X, X-x)$ (singular homology); (d) if one of the n dimensional groups in (c) is stable in the representation provided as inverse limit group, then so is the other.

(4) Finally, a number of results are proved concerning the relation between local homology invariants computed with singular and Vietoris homology, and their properties with regard to stability are examined. In particular it is shown that: (a) Generalized manifolds (using coefficients in a field) have stable local homology groups at every point; (b) if X is lc_s^q and lc_v^q, then singular $H_q(X, X-x) \approx$ Vietoris $H_q(X, X-x)$; (c) Vietoris $H_{r+1}(X, X-x) \approx$ Direct limit group [Vietoris H_r (neighborhoods of x with x deleted)]. In addition, there are some complicated theorems concerning local invariants. A footnote in proof states that S. Mardešić has proved that if X is paracompact Hausdorff, lc_s^n implies lc_v^n. [See Notices Amer. Math. Soc. 5 (1958), 210; Abstract 544–14.] This simplifies hypotheses in several of the theorems above.

{It should be remarked that S. T. Hu has obtained a method for defining local invariants without using inverse limits, that these invariants "give the same results" as the invariants of this paper for spaces suitably restricted, and he obtains easily some of the results of this paper, in particular, the "local Hurewicz theorem" [see S. T. Hu, Compositio Math. **13**, 173–218 (1958)]. On the other hand, for a map $f: X \to Y$, $f(x)=y$, unless the map f is such that $f^{-1}(y)$ is the single point x, the invariants of Hu are not functorial, a defect not shared by local invariants defined by inverse and direct limits in all cases.}

T. R. Brahana (Athens, Ga.)

873:

Wilder, R. L. Local orientability. Colloq. Math. **6** (1958), 79–93.

An n-dimensional generalised manifold S (an "n-gm") is called "locally orientable" if each point in S has an open neighbourhood U such that U is an n-gm. (The definitions are relative to a fixed field of coefficients.) Theorem: A necessary and sufficient condition that an n-gm be locally orientable is that every connected open subset be an n-gm. The author next gives an axiom system for n-gms, then modifies one axiom to a local version, say L, and calls the resulting system Δ_n. It is shown that Δ_n characterises the locally orientable n-gms, and nine further properties are each shown to be valid substitutes for L in the system Δ_n. The ninth of these involves chains of "canonical pairs" of neighbourhoods, and is like a global condition found by Begle, which in turn is related to Poincaré's definition for a combinatorial manifold. Further forms of Begle's global condition are obtained. The following "invariance of domain" theorem is also proved: a homeomorph, in a locally orientable n-gm S, of an n-gm, is open in S.

H. B. Griffiths (Bristol)

874:

Ganea, Tudor. Comment on the imbedding of polyhedra in Euclidean spaces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 27–32.

Recently A. Shapiro [Ann. of Math. (2) **66** (1957), 256–269; MR **19**, 671] and Wu Wen-Tsün [Bull. Acad. Polon. Sci. Cl. III **4** (1956), 573–577; MR **18**, 664] independently have found a necessary and sufficient condition for imbedding an n-complex X, $n \neq 2$, in Euclidean space E^{2n}, a problem initiated by van Kampen [Abh. Math. Sem. Hamburg **9** (1933), 72–78]. The condition is the vanishing of a certain characteristic class in $H^*(\tilde{X}^*/t)$ where $\tilde{X}^* = X \times X - \Delta$, Δ the diagonal and $t: X^* \to X^*$ is defined by $t(x, y) = (y, x)$. The author shows that this condition is equivalent to three other conditions on $X \times X$ or X^*. He also applies the work of van Kampen [ibid.] to answer a question of Marczewski [*Scottish Book*, Lwów, 1935–1941; transl. by S. Ulam, Los Alamos, 1957].

S. Smale (Princeton, N.J.)

875:

Lindsay, J. H., Jr. An elementary treatment of the imbedding of a graph in a surface. Amer. Math. Monthly **66** (1959), 117–118.

An obvious elementary proof that any finite connected graph may be imbedded in some orientable surface so as to form the vertices and boundaries (edges) of a map.

G. A. Dirac (Hamburg)

876:

Erdős, P. Graph theory and probability. Canad. J. Math. **11** (1959), 34–38.

Let $h(k, l)$ be the least integer so that every graph of $h(k, l)$ vertices contains either a closed circuit of k or fewer lines, or a set of l independent points. It is shown by a probabilistic argument that

$$h(k, l) > l^{1+1/2k};$$

and by a simple combinatorial argument that

$$h(2k+1, l) < cl^{1+1/k}, \quad h(2k+2, l) < cl^{1+1/k}.$$

Also discussed are bounds for the function $f(k, l)$, the least integer so that every graph of $f(k, l)$ vertices contains a complete graph of order k or a set of l independent points.

W. Moser (Winnipeg, Man.)

877:

Kotzig, Anton. Bemerkung zu den Faktorenzerlegungen der endlichen paaren regulären Graphen. Časopis Pěst. Mat. **83** (1958), 348–354. (Slovak. Russian and German summaries)

Bekanntlich [D. König, *Theorie der endlichen und unendlichen Graphen*, Akad. Verlag., Leipzig, 1936] lässt sich ein endlicher paarer regulärer Graph $m$ten Grades in m linear Faktoren zerlegen. Der Verf. legt das Problem vor, solche paare reguläre Graphen $m$ten Grades G $(m>2)$ aufzufinden, für welche eine Zerlegung in m lineare Faktoren $L_1, \cdots, L_m$ derart existiert, dass die Kompositionen $L_i \times L_j$ $(i \neq j)$ von je zwei dieser Faktoren Hamiltonsche Linien von G sind. Er beweist, dass eine notwendige Bedingung dafür ist, dass die Anzahl der Knotenpunkte von G durch 2, aber nicht durch 4 teilbar ist.

M. Fiedler (Prague)

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 689, 728, 730, 850, 868.

878:

★Strubecker, Karl. Differentialgeometrie. III. Theorie der Flächenkrümmung. Sammlung Göschen Bd. 1180/1180a. Walter de Gruyter and Co., Berlin, 1959. 254 pp. DM 4.80.

Vol. I of this text was devoted to the theory of curves

[1955; MR 16, 954]. Vol. II deals with those topics of surface theory which arise from a study of the first fundamental form alone [1958; MR 20 #4273]. This volume studies topics which arise when one also studies the second fundamental form.

Two sections are devoted to the theory of surface strips, then seventeen sections to the elementary theory of surface curvature, including familiar material like Meusnier's theorem, Euler's formula, Dupin's indicatrix, Rodrigues' formula, central surfaces. There follow twelve sections on topics related to the Gaussian curvature: Gauss-Bonnet integral formula, surfaces of constant Gaussian curvature, the problem of Minding on deciding whether two surfaces can be isometrically mapped. There are five sections on the Gauss-Codazzi equations, Bonnet's fundamental theorem of surface theory, and related topics. Lastly, there are six rather detailed sections on minimal surfaces.

Some noteworthy features are (i) the enrichment resulting from considering complex surfaces as well as real ones, as in the sections on the Monge and Serret surfaces with one family of curvature lines, the surfaces of rotation of constant curvature with isotropic axes of rotation, and the work on minimal surfaces; (ii) work in the large on convex surfaces; (iii) three fine sections on hyperbolic non-euclidean geometry tied in with work on surfaces of constant curvature; (iv) the detailed work on minimal surfaces.

The book is very well organized, clearly and carefully written. It is surprisingly detailed and thorough for its size and for that reason perhaps a little too concise at times. It uses a standard vector notation. An index, bibliography, illustrative examples, numerous historical credits are given, but there are no exercises for the student.

A. Schwartz (New York, N.Y.)

879:

Brauner, H. Über Strahlflächen von konstantem Drall. Monatsh. Math. **63** (1959), 101–111.

Ruled surfaces for which the parameter of distribution has the constant value zero are known to be developables. In this paper the author uses vector methods to investigate the properties of ruled surfaces for which the parameter of distribution has a constant non-zero value.

T. J. Willmore (Liverpool)

880:

Karapetyan, S. E. Harmonic quadrics and certain ruled surfaces of congruences. Dokl. Akad. Nauk SSSR **122** (1958), 335–338. (Russian)

Line congruences in three-space are studied by means of the coordinate tetrahedron $\{A_i\}$, with displacement given by $dA_i=\omega_i^k A_k$ $(i, k=1, 2, 3, 4)$. A family of the first order connected with the congruence is specialized by $\omega_1^4=0$, $\omega_2^3=0$. Interior differentiation gives equations of the kind $\omega_3^4=\alpha\omega_1^3-\beta\omega_2^4$, $\omega_1^2=\beta\omega_1^3+\gamma\omega_2^4$ and similar ones for ω_2^1 and ω_4^3: $\omega_2^1=\gamma'\omega_1^3+\beta'\omega_2^4$; $\omega_4^3=-\beta'\omega_1^3+\alpha'\omega_2^4$. The author, in this paper and in some previous ones [same Dokl. **117** (1957), 177–179; Naučn. Dokl. Vyš. Školy. Ser. Fiz.-Mat. **1, 2** (1958); MR **20** #302] has investigated ruled surfaces formed by the congruence, especially in the case that they are quadrics. These surfaces are here given by $\omega_2^4=\lambda\omega_1^3$. Harmonic ruled surfaces of the first and second focal surfaces of the congruence satisfy the equations $\alpha(\omega_1^3)^2-\gamma(\omega_2^4)^2=0$, $\alpha'(\omega_2^4)^2-\gamma'(\omega_1^3)^2=0$; for those of the first focal surface $\lambda=\varepsilon\sqrt{\alpha/\gamma}$, $\varepsilon=\pm 1$. A number of theorems are found for the case that these surfaces are quadrics, and some other noteworthy ruled surfaces of the congruence are also discussed.

D. J. Struik (Cambridge, Mass.)

881:

Ščerbakov, R. N. Projective theory of reference frames of a ruled surface belonging to a given congruence. Mat. Sb. N.S. **46** (88) (1958), 159–194. (Russian)

Par les méthodes de Cartan on construit le repère canonique de la congruence de droites dans S_3 qui contient un réseau conjugué des surfaces réglées donné d'avance. Cela nous donne un repère canonique d'une surface plongée dans la congruence; on trouve les significations géom. de tous les invariants et on étudie les classes spéciales des surfaces d'une congruence ainsi que les congruences qui les contiennent.

A. Švec (Prague)

882:

Vincensini, Paul. Sur une représentation dans E_4 des congruences W à nappes focales réglées de E_3. Arch. Math. **9** (1958), 360–365.

The purpose of this paper is to study the W congruences with two ruled focal surfaces and a given ruled surface as one of their focal surfaces in ordinary projective space by means of a representation of lines in an ordinary Euclidean space E_3 into the Euclidean space E_4 of four dimensions [for this representation see P. Vincensini, Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40** (1954), 1090–1105; Univ. e Politec. Torino. Rend. Sem. Mat. **14** (1954–55), 311–323; MR **16**, 955; **17**, 1233]. It is shown that with each of the two focal surfaces of such a W congruence there exist a one-parameter family of linear congruences having the second order contact and a one-parameter family of linear complexes having the third order contact. Moreover, the two flecnode ruled surfaces of any ruled surface are the focal surfaces of a W congruence.

C. C. Hsiung (Bethlehem, Pa.)

883:

Terracini, Alessandro. Sui sistemi F di linee spaziali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 220–226.

L'Autore considera i sistemi F di linee dello S_3 proiettivo, ossia i sistemi ∞^6 di linee $y=y(x)$, $z=z(x)$ integrali di un sistema di equazioni differenziali della forma:

$$y''' = y''(Ay''+Bz'')+C_1y''+D_1z''+E_1$$
$$z''' = z''(Ay''+Bz'')+C_2y''+D_2z''+E_2$$

(con $A, B, C_1, D_1, E_1, C_2, D_2, E_2$ funzioni di x, y, z, y', z'), e dimostra che un sistema ∞^6 di linee definito dal sistema di equazioni differenziali: $y'''=\varphi(x, y, z, y', z', y'', z'')$, $z'''=\psi(x, y, z, y', z', y'', z'')$ (con φ, ψ soddisfacenti ad opportune condizioni di regolarità) è un sistema F se e solo se i tre E_3 delle tre linee del sistema contenenti tre E_2 uscenti da uno stesso punto con la medesima tangente ed appartenenti ad una calotta del second'ordine σ_2^2, appartengono necessariamente ad una calotta del terz'ordine σ_2^3.

Inoltre l'A. determina tutti i sistemi F spaziali costituiti di curve appartenenti tutte a complessi lineari, in modo tale che questi complessi appartengano ad un sistema lineare della dimensione minima possibile, e cioè ad una rete di complessi lineari.

D. Gallarati (Genoa)

884:

★**Lagally, Max. Vorlesungen über Vektorrechnung.** 6te Aufl. bearbeitet von W. Franz. Mathematik und ihre Anwendungen in Physik und Technik. Reihe A, Bd. 2. Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1959. xxii+462 pp. DM 22.00.

The 5th edition [1956] was reviewed in MR **19**, 574. The 6th edition is almost entirely unchanged.

885:

de Mira Fernandes, A. Estensori jacobiani parziali e derivati. Univ. Lisboa. Revista Fac. Ci. A (2) **6** (1957/58), 147–156.

H. V. Craig [H. V. Craig and W. T. Guy, Jr.: Amer. J. Math. **72** (1950), 229–246; MR **11**, 543] has introduced the concept 'Jacobian extensor', to which the present author added 'Jacobian pseudo-tensor' [Univ. Lisboa. Revista Fac. Ci. A (2) **5** (1956), 249–264; MR **19**, 455]. Here the concept is generalized to 'partial Jacobian' (or 'complementary Jacobian'), by means of which one can define the 'Jacobian extensor' utilizing a known property of determinants. (The idea consists in replacing the variables by their partial derivatives.) Peculiar notations of Craig are adopted. The terms 'partial Jacobian extensors', 'derived Jacobian extensors' and 'Hessian extensors' are introduced.

T. Takasu (Yokohama)

886:

Craig, Homer V. On primary extensors. Tensor (N.S.) **8** (1958), 196–206.

The primary extensors relative to a function $F(x, x')$ of the coordinates x^a and their derivatives x'^a with respect to a parameter t are defined as the extensors $F_{;\alpha\beta} = \partial F/\partial x^{(\alpha)a}$ and $F_{\alpha\beta} = T_a^{(1-\alpha)}$ $(\alpha = 0, 1, \cdots, M)$, subject to the stipulations $M = 1$, and T_a is the non-vanishing tensor member of $F_{;\alpha\beta}$. There is also presented a definition of the primary extensors associated with a function $f(x, y_a)$ of the x^a and the components y_a of a covariant tensor, and it is then proved that if $F(x, x')$ and $f(x, y_a)$ are related by a certain Legendre transformation, then their primary extensors are the same. An immediate conclusion is that the Lagrangian and Hamiltonian equations each express the equality of the same pair of extensors. Examining the first variation of the kinetic-potential expressed as a Legendre transform of the Hamiltonian total energy $h(q, p)$, it is found that it is expressible in two different ways as extensor constructions, one of which leads to the extensor construction of the Lagrangian brackets. In consideration of the possibility of altering the key functions for the Lagrangian equations of motion, one finds that any two linearly independent combinations of kinetic and potential energy may be used as key functions. Finally it is noted in the case of calculus of variations problems with added conditions of the type $G(x, x') = 0$, that $G(x, x')$ is not necessarily indicated as a key function but instead a function of the type $g(t)G(x, x')$ with only the very slight restriction that $g(t)G(x, x') = 0$ and $G(x, x') = 0$ define the same set of admissible arcs. *A. Kawaguchi* (Sapporo)

887:

Moór, Arthur. Über die kovariante Ableitung der Vektoren. Acta Sci. Math. Szeged **19** (1958), 237–246.

The idea of the covariant derivative of a geometric object which was first formulated by Schouten was given an explicit form in the one-dimensional case by S. Gołab [Nieuw Arch. Wiskunde (3) **2** (1954), 90–96; MR **16**, 76], who also gave a slightly more general formulation than Schouten. In this paper the author gives the definition of the covariant derivative of vectors corresponding to the more general formulation given by S. Gołab, which involves four conditions. With the addition of a fifth condition the explicit form of the covariant derivatives of vectors in n-space are given.

E. T. Davies (Southampton)

888:

Ahn, Jae-koo. On the projective and conformal transformations in the metric manifold with torsion. Kyungpook Math. J. **1** (1958), 23–32.

This paper is based on the classical treatments of projective and conformal geometry such as appear in the two books by Eisenhart [*Riemannian geometry*, Princeton Univ. Press, Princeton, N.J., 1926; *Non Riemannian geometry*, Amer. Math. Soc., New York, 1927]. The author makes an incorrect statement in Section II that the coefficients of affine connection, when contracted with respect to two of the three indices, lead to the components of a covariant vector. This leads him to an incorrect theorem on the Jacobian of a coordinate transformation.

E. T. Davies (Southampton)

889:

Whitney, H.; et Bruhat, F. Quelques propriétés fondamentales des ensembles analytiques-réels. Comment. Math. Helv. **33** (1959), 132–160.

In recent years real analytic manifolds have become the object of deep study. The theory of Stein manifolds was transposed in the case of real analytic sets by H. Cartan [Bull. Soc. Math. France **85** (1957), 77–99; MR **20** #1339]. The present paper is a systematic study of C-analytic sets, i.e., the real trace of complex analytic sets.

In the first section the authors give a proof of the existence and a kind of uniqueness of the complexification Ω^* of a real analytic manifold Ω. This was proved by several authors independently. Then come some discussions of proper coordinate systems of an analytic set in the complex and real cases. The following fact is shown (proposition 3 in § 4). Let Ω^* be a complexification of a real analytic manifold A, and E^* be a complex analytic subset in Ω^*. Put $E = A \cap E^*$. There exists an open neighborhood V of a point a in A such that for every neighborhood V^* of V in Ω^*, every irreducible component of $E^* \cap V^*$ in V^* which meets V passes through the point a.

Next the notion of A-germs is introduced. Consider a pair (V^*, E^*) of an open neighborhood V^* of A in Ω^*, and a complex analytic subset E^* in V^*. Two such pairs (V_1^*, E_1^*) and (V_2^*, E_2^*) are said to be A-equivalent if there exists a neighborhood V_3^* of A in Ω^* such that $E_1^* \cap V_3^* = E_2^* \cap V_3^*$. A class of (V^*, E^*) classified by A-equivalence is called an A-germ. The collection of all A-germs is denoted by $\mathfrak{G}(A)$. On the other hand, as was shown in Séminaire H. Cartan École Norm. Sup. 1953/54, Exp. VI, VII [MR **19**, 577], we have a topological space $G(\Omega^*)$ consisting of all irreducible germs of complex analytic sets in Ω^*. The topology of $G(\Omega^*)$ is separated (T_2-space), locally compact and locally connected. Let π be the canonical projection of $G(\Omega^*)$ onto Ω^*, which assigns the point a to every germ at a. For every element $Y \in \mathfrak{G}(A)$, every representation (V^*, E^*) of Y gives rise

to a definite set $\Gamma(Y)$ in $G(A)=\pi^{-1}(A)$ by $\Gamma(Y)=\Gamma(V^*, E^*)\cap\pi^{-1}(A)$, where $\Gamma(V^*, E^*)$ is the collection of all irreducible germs given by E^* at all points in V^*. The following fundamental theorem (proposition 6 in § 5) is proved: For an element $Y\in\mathfrak{G}(A)$ and a point $x\in\pi(\Gamma(Y))$, there exists an open neighborhood U of x in A such that, if a germ at the point x of $Z\in\mathfrak{G}(U)$ contains the germ of Y at the point x, then Z contains the canonical image of Y in $\mathfrak{G}(U)$. This means a minimal property of a C-analytic germ E in a fixed neighborhood of an arbitrary point of E.

After some discussion of the notions of intersections, unions and irreducible components of A-germs, a C-analytic set E in a real analytic manifold Ω is defined as a set $E=\pi(\Gamma(Y))$ for a suitable element $Y\in\mathfrak{G}(\Omega)$. This definition is equivalent to the following: (a) There exists a neighborhood V^* of Ω in Ω^* and a complex analytiz subset E^* in V^* such that $E=E^*\cap\Omega$; (b) E is the set of zero points of a coherent sheaf of ideals on Ω; (c) E is the set of zero points of a finite number of (in reality we can replace it by only one) real analytic functions on Ω. Though every real-analytic set is C-analytic locally, this is not true in the large. An example is given in the final section of the present paper: the union of a line $x=y=0$ and the set

$$x^2+2(1+\sin z)xy+y^2=0 \quad (2k\pi\leqq z\leqq(2k+1)\pi),$$

in the space R^3. Other examples are given by F. Bruhat and H. Cartan [C. R. Acad. Sci. Paris **244** (1957), 988–990, 1123–1126; MR **19**, 125, 535]. In the remaining sections, the authors discuss some properties of C-analytic sets, such as decompositions into C-irreducible components, the notions of C-dimensions and C-ranks. In the final section (§ 11), several interesting examples are given, which illustrate the complicated features of general real analytic sets. They also suggest the distinctions between C-analytic sets, general real analytic sets and real algebraic varieties.

S. Hitotumatu (Tokyo)

890:

Frölicher, Alfred; and Nijenhuis, Albert. Theory of vector-valued differential forms. II. Almost-complex structures. Nederl. Akad. Wetensch. Proc. Ser. A **61** = Indag. Math. **20** (1958), 414–429.

The theory of vector-valued differential forms developed by the authors in part I [same Proc. **59** (1956), 338–359; MR **18**, 569] is extended here to take account of almost-complex structures. A large number of identities are established.

M. F. Atiyah (Cambridge, England)

891:

Ishihara, Shigeru. Holomorphically projective changes and their groups in an almost complex manifold. Tôhoku Math. J. (2) **9** (1957), 273–297.

The paper consists of two parts; the first containing a discussion of relations between the torsion of ϕ-connections in manifolds with almost-complex structure ϕ; the torsion of the tensor ϕ and other related quantities. This part is, in general, a continuation of Obata's work [Jap. J. Math. **26** (1956), 43–77; MR **20** #1796a], but distinguishes itself by its emphasis on half-symmetric (new concept) and semi-symmetric connections. Into this pattern fit the holomorphically projective changes of half-symmetric ϕ-connections, which are those changes that leave unchanged the curves along which the plane spanned by the tangent vector and its ϕ-transform is covariant constant (holomorphically planar curve).

The second part deals with groups of diffeomorphisms of the manifold which leave ϕ invariant and effect a holomorphically projective change of the connection (H-projective transformation), or leave invariant the Ricci tensor. Such a group is called "essentially affine" if there is a holomorphically related half-symmetric ϕ-connection which is invariant under the group. In that case the group is a subgroup of the group of affine transformations of that connection. Groups of H-transformations turn out to be essentially affine if the isotropy group is compact. The same happens if the group is transitive; the connection is not H-projectively flat and the identity component of the linear isotropy group is irreducible at one point. These theorems lead to conditions under which the groups of affine transformations, of H-projective transformations, isometries, and their variations (including those leaving invariant the Ricci tensor) of a Kaehler manifold are equal. The conditions include properties of the various holonomy groups.

A. Nijenhuis (Seattle, Wash.)

892:

Gaffney, Matthew P. The conservation property of the heat equation on Riemannian manifolds. Comm. Pure Appl. Math. **12** (1959), 1–11.

Let M be a Riemannian manifold, $P\in M$ and let r be the distance from P. If M has the property that $e^{-\alpha r^2}$ is integrable on M for all positive α, then the author proves the following conservation property. Let u be a solution of the heat equation $\delta_e\kappa d'u=-\sigma(\partial u/\partial t)$ with a compact carried initial function f. Then $\int_M u\sigma dV=\int_M f\sigma dV$ for all t. In the above δ_e denotes the negative of the divergence operator restricted to C^1 forms with compact carriers, d' is the Hilbert space adjoint of δ_e and $\bar{\delta}_e$ is the closure of δ_e.

J. J. Kohn (Waltham, Mass.)

893:

Hwang, Cheng-chung. On the isometric correspondence of Riemann spaces of category $n-1$. Advancement in Math. **4** (1958), 123–126. (Chinese. English summary)

A Riemannian space is called of category p if it has p functionally independent scalar invariants and no more. The author gives a necessary and sufficient condition for two n-dimensional Riemannian spaces of category $n-1$ to be locally isometric. As mentioned by the author, this problem has been studied by J. T. Sun [Duke Math. J. **16** (1949), 571–573; MR **11**, 399], but the proof therein is incomplete.

H. C. Wang (Evanston, Ill.)

894:

Stavroulakis, Nicias. Généralisation de la notion de nappe logarithmique. Cylindres logarithmiques. Généralisations de la notion de point logarithmique. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 529–548.

In two previous papers [C. R. Acad. Sci. Paris **246** (1958), 1149–1152, 1368–1371; MR **20** #2749a, b] the author examined certain types of singularities on 2-dimensional surfaces. The present paper shows how certain notions introduced in those papers can be generalized.

T. J. Willmore (Liverpool)

895:

Eum, Sang-Seup. Notes on Kaehlerian metric. Kyungpook Math. J. **1** (1958), 13–21.

Dans un n-espace analytique complexe hermitien (coordonnées z^α, $z^{\bar{\alpha}}=\overline{z^\alpha}$; métrique $g_{\alpha\bar{\beta}}=\overline{g_{\beta\bar{\alpha}}}$), l'existence

d'un système orthonormé $[\eta_{p|}]$ de n vecteurs a été montrée par T. Suguri [Mem. Fac. Sci. Kyūsyū Univ. A **6** (1951), 31–40; MR **14**, 86]; l'auteur montre ici (Section 2) que si l'on peut trouver un système $[\eta_{p|}]$ qui soit analytique complexe, la condition pour que l'espace soit kählérien est équivalente à l'une des suivantes: (B) $\eta_{p|}{}^{\alpha}$; $\eta_{p|\alpha}$ sont parallèles par rapport à la métrique; (C) un groupe de transformations engendré par $x_p f = \eta_{p|}{}^{\alpha}(\partial f/\partial z^{\alpha})$ est abélien. Si $[\eta_{p|}]$ n'est pas analytique complexe il est suffisant, pour que l'espace soit kählérien, que: $\eta_{t|\overline{p}}\eta_{s|}{}^{\beta} = \eta_{s|\overline{p}}\eta_{t|}{}^{\beta}$.

Dans les sections 3 et 4 sont étudiés les espaces kählériens à courbure holomorphique constante: (Section 3) si l'espace kählérien est de Fubini, i.e.,

$$g_{\alpha\bar{\beta}} = \frac{\partial^2\phi}{\partial z^{\alpha}\partial\bar{z}^{\beta}} \quad \text{où} \quad \phi(z,\bar{z}) \equiv f(p); \quad p = \sum_{\alpha=1}^{n} z^{\alpha}\bar{z}^{\alpha};$$

il est à courbure holomorphique constante si et seulement si

$$f(p) = \frac{1}{a}\log(ap+b)+c$$

($a \neq 0$, $b \neq 0$, c sont des constantes).

Ce résultat prolonge celui de Bochner [Bull. Amer. Math. Soc. **53** (1947), 179–195; MR **8**, 490]. (Section 4) L'auteur établit une relation $N_{\alpha\bar{\beta}\gamma\bar{\delta}}(R_{\alpha\bar{\beta}\gamma\bar{\delta}}, g_{\alpha\bar{\beta}}) = 0$ nécessaire et suffisante pour qu'un espace kählérien soit à courbure holomorphique constante. *J. Renaudie* (Rennes)

896:

Vranceanu, G. Spazi a connessione affine e le algebre di numeri ipercomplessi. Ann. Scuola Norm. Sup. Pisa (3) **12** (1958), 5–20.

A constant affine connection P is a connection whose components $\Gamma_{jk}{}^{i}$, in an appropriate system of coordinates, are constant: this happens if and only if the affine space A_n determined by the connection admits a simply transitive and abelian group of transformations.

The link between A_n and an algebra $\mathfrak{A}_n$ of hypercomplex numbers $x = e_i x^i$ is given by the multiplication rule $e_j e_k = \Gamma_{jk}{}^{i} e_i$. Properties concerning $\mathfrak{A}_n$ can be expressed as geometric properties of A_n; a number of consequences following from this interpretation are derived. *E. Bompiani* (Rome)

897:

Murgescu, Viorel. Les invariants d'un champ tensoriel dans les espaces à parallélisme absolu. Bul. Inst. Politehn. Iaşi (N.S.) **3** (1957), 29–36. (Romanian. Russian and French summaries)

Let A_n be a space of affine connection with coefficients $\Gamma_{jk}{}^{i}$ which are in general non-symmetric in j and k, and τ_{jk} a doubly covariant tensor field in A_n. It is assumed in addition that the space A_n is flat, i.e., its curvature tensor $R^{h}_{ijk} = 0$. The author considers the problem of finding a function $\mathcal{F}(x^i, \tau_{jk})$ which remains invariant under the parallel transport of the tensor τ_{jk}. This leads to a system of linear partial differential equations, where the independent variables are the x^i and τ_{jk}, which is completely integrable because of $R^{h}_{ijk} = 0$. The integration of this system is achieved by means of a transformation of the form $\bar{\tau}_{pq} = \lambda_p{}^{j}\lambda_q{}^{k}\tau_{jk}$ in which the $\lambda_p{}^{j}$ are n linearly independent contravariant and parallel vector fields in A_n, the existence of which is also assured by the condition $R^{h}_{ijk} = 0$. The function $\mathcal{F}$ is then independent of x^i and the $\bar{\tau}_{pq}$ form, therefore, a complete system of invariants.

Finally the author considers the particular case $\tau_{jk} = 0$ ($j \neq k$) and gives a recursion formula for its n invariants. *R. Blum* (Saskatoon, Sask.)

898:

Murgescu, Viorel. Sur les espaces à connexion affine, à courbure récurrente. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **4** (1958), 67–79. (Romanian and Russian summaries)

A space with affine (in general non-symmetric) connection A_n is called (a) curvature recurrent if $R^{h}_{ijkl} = \varphi_l R^{h}_{ijk}$, and (b) Ricci-recurrent if $R_{ij|l} = \varphi_l R_{ij}$. These definitions are in analogy with curvature recurrent and Ricci-recurrent riemannian spaces introduced by E. M. Patterson [J. London Math. Soc. **27** (1952), 287–295; MR **14**, 88] and C. M. Chaki [Rend. Sem. Mat. Univ. Padova **26** (1956), 168–176; MR **18**, 821]. R^{h}_{ijk} and R_{ij} are, of course, the curvature tensor and Ricci tensor respectively. The index after the vertical bar | indicates covariant differentiation and φ_l is the so-called 'recurrence vector'. It is obvious that (a) implies (b) but not inversely.

After deriving a number of properties of curvature recurrent and Ricci-recurrent A_n, the following two main theorems are established.

(1) The necessary and sufficient condition for a curvature recurrent or Ricci-recurrent A_n to admit, for every vector $\overline{V}(v^i)$, an invariant function of the form $\Lambda = \sqrt{(\varepsilon^{-\lambda} E_{ij} v^i v^j)}$ called the 'length' of $\overline{V}$ is that the recurrence vector φ_l should be the gradient of a scalar function.

(2) Every curvature recurrent or Ricci-recurrent A_n admits an angular metric; i.e., an invariant function attached to every two directions (X^i) and (Y^i), of the form

$$\cos V = \frac{E_{ij}X^iY^j}{\sqrt{(E_{ij}X^iX^j)}\sqrt{(E_{ij}Y^iY^j)}}.$$

E_{ij} is the 'symmetrized' Ricci tensor: $E_{ij} = \frac{1}{2}(R_{ij} + R_{ji})$. *R. Blum* (Saskatoon, Sask.)

899:

Hlavatý, Václav. The holonomy group. I. The curvature tensor. J. Math. Mech. **8** (1959), 285–307.

This paper is the introductory part of a systematic study of over-determined systems of partial differential equations and their solutions obtained by algebraic methods. It deals with two problems as follows: Given a metric tensor ${}^{\circ}g_{\lambda\mu}$ with the Christoffel symbols ${}^{\circ}\Gamma^{\nu}_{\lambda\mu}$ one has to find (1) a tensor $Q_{\lambda\mu}{}^{\nu}$ (symmetric in λ and μ) for which $\Gamma^{\nu}_{\lambda\mu}$, defined by $\Gamma^{\nu}_{\lambda\mu} = {}^{\circ}\Gamma^{\nu}_{\lambda\mu} + Q_{\lambda\mu}{}^{\nu}$, are Christoffel symbols and (2) the tensor $g_{\lambda\mu}$ whose Christoffel symbols are $\Gamma^{\nu}_{\lambda\mu}$. Any solution $Q_{\lambda\mu}{}^{\nu}$, $g_{\lambda\mu}$ of this problem is obtained by elementary algebraic methods.

A. Kawaguchi (Sapporo)

900:

Scagliotti, Lucia. Sulla geometria intrinseca dei gruppi continui finiti a tre parametri di omografie piane. Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956–57), 439–445.

Si considerano i vari tipi di gruppi continui finiti a tre parametri di omografie piane per i quali è applicabile la teoria di Pick-Kowalewski la quale permette di associare a ciascun gruppo un elemento d'arco.

Si dimostra che tra i detti gruppi il gruppo G_3 che muta in se un dato elemento differenziale del terzo ordine E_3 è l'unico che ammetta geodetiche ben determinate che non siano rette. *C. Longo* (Parma)

901:

Speranza, Francesco. Sulle trasformazioni che posseggono un gruppo di coppie di corrispondenze in sè. I. Boll. Un. Mat. Ital. (3) **13** (1958), 486–496. (English summary)

Let F and $\bar{F}$ be two manifolds, Ω [$\bar{\Omega}$] be a mapping of F [$\bar{F}$] into itself, and T a transformation of F into $\bar{F}$. If $\bar{\Omega}T = T\Omega$, the pair $(\Omega, \bar{\Omega})$ is called a "couple of mappings relative to T". Suppose that Ω_i and $\bar{\Omega}_i$ are corresponding elements of two fixed isomorphic groups g and $\bar{g}$ acting on F and $\bar{F}$, respectively. The problem is to find those T such that all pairs $(\Omega_i, \bar{\Omega}_i)$ are couples of mappings relative to T. A general construction for T is given and various properties are derived.

C. B. Allendoerfer (Seattle, Wash.)

902:

Adler, Alfred. A characteristic relation in fibre bundles. Amer. J. Math. **79** (1957), 713–724.

Let (M, X, F, G, p) be a fibre bundle where bundle space M and base space X are Riemannian spaces and fibre F is a homogeneous space of the group G. S. S. Chern posed the following problem. What are the Pontrjagin characteristic classes of M in terms of those of X and F?

The author gives an answer to this problem in the case where the fibre F is of the form G/K, G a compact Lie group and K a closed connected subgroup of G. In the first part of the paper, he finds the curvature form of M in terms of those of X and F and in the second part he inserts this curvature form in the Pontrjagin polynomials of M. Then the Pontrjagin polynomials split into tensor products and give the answer. *K. Yano* (Tokyo)

903:

Reinhart, Bruce L. Line-element fields on the torus. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 49–50.

Let F be a line element field on a 2-torus T^2, that is, a C^1-map $F: T^2 \to S^1$. Let $F_*: H_1(T^2) \to H_1(S^1)$ be the induced homology map. For each closed curve $\mathscr{C}$ on T^2 lying in homology class $\mathscr{C}^*$, $F_*(\mathscr{C}^*) = \mu\xi$, where ξ is a generator of $H_1(S^1)$ and μ is the winding number of $\mathscr{C}$ with respect to F.

The author shows that if there exists on T^2 a curve whose winding number with respect to F is non-zero, then there exists a closed integral curve of F.

Let Γ and Γ_1 form a basis for $H_1(T^2)$ and set $F_*(\Gamma) = i\xi$, $F_*(\Gamma_1) = j\xi$. Then the number of closed integral curves is $\geq$ g.c.d. $[i, j]$.

L. Markus (Minneapolis, Minn.)

904:

Tihomirova, E. S. A new proximity invariant. Uspehi Mat. Nauk **13** (1958), no. 5 (83), 197–202. (Russian)

The author calls a Riemannian space R infinitesimally acyclic in dimension r if there exist (large) constants c and C such that every singular r-cycle on a set of diameter $d > c$ is the boundary of a singular cell on a set of diameter at most Cd. (Compact $\to$ acyclic in this sense. The term "infinitesimal" is used because the property is invariant under infinitesimal transformations = uniform equivalences.) A typical non-acyclic space is the paraboloid $z = x^2 + y^2$. More generally, the author defines a paraboloid as a surface $z = F(x_1, \cdots, x_n)$, F a polynomial with a unique critical value at the origin. Theorem: the paraboloid is infinitesimally acyclic in dimension r if and only if the $(r+1)$-dimensional type numbers of F and of $-F$ both vanish. For quadratic F, the author obtains a complete uniform classification; with some exceptions uniformly equivalent to E^n, uniform equivalence implies affine equivalence. Finally, she shows that a Riemannian space having spherical symmetry is infinitesimally acyclic in dimensions other than $n-1$.

J. Isbell (Seattle, Wash.)

905:

Bakel'man, I. Ya. Definition of a convex surface by a given function of its principal curvatures. Dokl. Akad. Nauk SSSR **123** (1958), 215–218. (Russian)

Let D be a closed convex domain in E^2 bounded by the closed curve C. The paper deals with the existence of convex surfaces $z(x, y)$ defined in D, within certain classes W_1, W_2 discussed below, satisfying either of

(1) $f(x, y, z, p, q)K - \phi(x, y, p, q)H = \psi(x, y)$

(2) $f(x, y, z, p, q)K - \phi(x, y, p, q)f^k(x, y, z, p, q)H = \psi(x, y)$,

where K, H are the Gauss and mean curvature of $z = z(x, y)$, f and ϕ are positive and continuous in all variables, f_z is non-positive and continuous, and ψ is summable over D.

Let a space curve L with a 1-1 projection on C be given. The boundary of the convex closure of L is decomposed by C into two developable surfaces; let T be the one below L. For any convex surface z defined in D we put (as in the author's previous paper, Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **13** (1958), no. 1, 25–38 [MR **20** #3384]),

$$w(z, f, M) = \iint_M fK\,dx\,dy, \quad h_1(z, \phi, M) = \iint_M \phi H\,dx\,dy,$$

$$h_2(z, \phi, M) = h_1(z, \phi\sqrt{f}, M),$$

where M is a Borel set in D. Then $w(T, f, M) = 0$, $h_i(T, \phi, M) \geq 0$. Let $\psi_i(M)$ be totally additive set functions in D satisfying $-h_i(T, \phi, M) \leq \psi_i(M)$. The class W_i consists of all convex surfaces, u defined in D, convexity downward, lying under T and such that: (1) $w(u, f, M) - h_i(u, \phi, M) \leq \psi_i(M)$ for all Borel $M \subset D$; (2) W_i does not contain u, v with $u - v = \text{const.}$; (3) $z = u(x, y)$ contains points of L.

For $u \in W_i$ denote by $v(u)$ the volume bounded by the surfaces u, T and the cylinder parallel to the z-axis with generators through C, and put $v_i = \sup_{u \in W_i} v(u)$. If $v_i < \infty$, then there are $z_i \in W_i$ realizing this maximum, and if $\psi_i(M)$ is absolutely continuous, then z_i is a (generalized) solution of the differential equation i $(i = 1, 2)$. Conditions under which $v_i < \infty$ are given, for example: If $f_0(p, q)$ is summable and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_0(p, q)(1 + p^2 + q^2)^{-2}\,dp\,dq < \infty$; moreover, if $f(x, y, z, p, q) \leq f_0(p, q)$ and $\phi(x, y, p, q) \geq k > 0$; then $v_1 < \infty$. Similarly, $v_2 < \infty$ if $\phi\sqrt{f} \geq k > 0$.

H. Busemann (Cambridge, Mass.)

906:

Aleksandrov, A. D. Uniqueness theorems for surfaces in the large. II. Vestnik Leningrad. Univ. **12** (1957), no. 7, 15–44. (Russian. English summary)

[For part I see Vestnik Leningrad. Univ. **11** (1956), no. 19, 5–17; MR **19**, 167.] Let R be an $(n+1)$-dimensional Riemann-space with coordinates $x^1, \cdots, x^{n+1}$. In R consider a set Σ of surfaces of the form $x^{n+1} = z(x^1, \cdots, x^n) = z(x)$, and call Σ convex if, with $z^0(x)$ and $z^1(x)$, the surfaces $(1-t)z^0(x) + tz^1(x)$, $0 \leq t \leq 1$, are also in Σ. The principal

curvatures of $z(x)$ (when they exist) are denoted by $k_1 \geqq k_2 \geqq \cdots \geqq k_n$; the sign of the k_i at (x, z) is determined by choosing the $(n+1)$st coordinate vector such that it forms with the normal of the surface an angle $\leqq \pi/2$.

Let $\phi(k_1, \cdots, k_n, z_1, \cdots, z_n, z, x^1, \cdots, x^n) = \phi(k_i, z_i, z, x)$ ($z_i = \partial z/\partial x^i$ and (later) $z_{ik} = \partial^2 z/\partial x^i \partial x^k$) be of class C^1 in k_i, z_i, z for fixed x. Let ϕ be defined on all s of a convex set Σ. If for two surfaces $S^0, S^1 \in \Sigma$ given by $z^0(x)$, $z^1(x)$ at every point x, where both functions are defined, $\phi(k_i^0, z_i^0, z^0, x) = \phi(k_i^1, z_i^1, z^1, x)$, then the difference $\Delta z = z^1 - z^0$ satisfies at these points a differential equation

$$A^{jk} z_{jk} + B^r \Delta z_r + C \Delta z = 0, \tag{1}$$

where the coefficients A^{jk}, B^r, C are functions of x determined by ϕ, z^0, z^1. If all $\partial\phi/\partial k_i$ have the same sign, then (1) is elliptic. For a set, where z_i^0, z_i^1 are bounded and $H > \partial\phi/\partial k_i > h > 0$, there are positive constants A, a such that $A\Sigma\xi_i^2 > A^{jk}\xi_j\xi_k \supset a\Sigma\xi_c^2$.

Even when ϕ, z^0, z^1 are analytic, the coefficients A^{jk}, B^r, C are, in general, not continuous. However, when S^0, S^1 are of class C^2 and ϕ, $\partial\phi/\partial k_i$, $\partial\phi/z_i$, $\partial\phi/\partial z$ are continuous in all arguments, and if $\partial\phi/\partial k_i = \partial\phi/\partial k_j$ whenever $k_i = k_j$, then the coefficients in (1) are continuous functions of x. *H. Busemann* (Cambridge, Mass.)

907:

Aleksandrov, A. D. Uniqueness theorems for surfaces in the large. III. Vestnik Leningrad. Univ. **13** (1958), no. 7, 14–26. (Russian. English summary)

As in part II consider a hypersurface S of the form $x^{n+1} = z(x^1, \cdots, x^n)$ in an $(n+1)$-dimensional Riemann space. Instead of the z_i consider the unit normal n of S at (x, z) and a function $\phi(s, x) = \phi(k_1, \cdots, k_n, n, x)$, where it is now assumed that ϕ is of class C^1 for each fixed x in the remaining variables, and that these derivatives are bounded in every compact set of all variables, moreover, $\partial\phi/\partial k_i \geqq 0$, and locally $\partial\phi/\partial k_i > \beta > 0$. The main tool is the following variation of the principal theorem in II.

(*) Let S^0, S^1 be surfaces given by $z = z^0(x)$, $z = z^1(x)$, and let S^t be the surface $(1-t)z^0(x) + tz^1(x)$, $0 \leqq t \leqq 1$. Let ϕ be defined for all S^t. Then

$$\Delta\phi = \phi(S^1, x) - \phi(S^0, x) = A^{ik}\Delta z_{ik} + B^i \Delta z_i + C\Delta z,$$

where z_i, z_{ik} are partials, $\Delta z = z^1(x) - z^2(x)$, etc. $A^{ik}(x)$, $B^i(x)$ and $C^i(x)$ are bounded, and $A^{ik}\xi_i\xi_k$ is positive definite. Then $A^{ik}\xi_i\xi_k > a\sum \xi_i^2$. For this conclusion it suffices that $\partial\phi/\partial k_i > \beta > 0$ holds either on S^0 or S^1.

Let $\bar{S}$ be a smooth surface contained in a domain covered by a family of surfaces S, where $\bar{S}$ is defined in terms of the same coordinates as S by an equation $z = \bar{z}(x_1, \cdots, x_n)$. The normal to $\bar{S}$ is also oriented towards increasing z. Let ϕ be defined and have all the listed properties for all surfaces $z = (1-t)\bar{z}(x) + tz(x)$, $0 \leqq t \leqq 1$. If $S^1 \in \{S\}$ exists, touching $\bar{S}$ from the side of increasing z, and if a connected component C of points of contact exists, such that the conditions of (*) are satisfied in the neighborhood of every point of C, then $\bar{S} \subset S^1$. (An analogous theorem holds, of course, when the contact is from the side of decreasing z.)

This theorem contains various theorems of the author and others as special cases or corollaries, for example, it characterizes spheres in E^n or in spaces of constant curvature if $\{\bar{S}\}$ is a family of spheres with the same center z. For instance, let ψ be a function like ϕ, but depending only on $k_1 \geqq k_2 \geqq \cdots \geqq k_m$, $m \leqq n$. If $\psi \equiv 1$ on all spheres S, $\psi(\bar{S}, x) \leqq 1$ on $\bar{S}$ and $\bar{S}$ is a closed surface, then $\bar{S}$ is a sphere, provided it satisfies the conditions of (*) at the points with maximal distance from z.

H. Busemann (Cambridge, Mass.)

908:

Aleksandrov, A. D.; and Volkov, Yu. A. Uniqueness theorems for surfaces in the large. IV. Vestnik Leningrad. Univ. **13** (1958), no. 13, 27–34. (Russian. English summary)

Theorems very similar to those of the preceding paper are proved for relative differential geometry, where the euclidean unit sphere is replaced by a symmetric convex surface E with positive curvature. For example, the relative spheres λE are characterized by conditions on a function $\phi(k_1, \cdots, k_n, n, z)$ (see above), where $k_1, \cdots, k_n$ are now the relative principal curvatures and n is the relative normal, i.e., we compare points of the surface considered with points of E where the tangent planes are parallel. *H. Busemann* (Cambridge, Mass.)

909:

Aleksandrov, A. D. Uniqueness theorem for surfaces in the large. V. Vestnik Leningrad. Univ. **13** (1958), no. 19, 5–8. (Russian. English summary)

Let $\phi(k_1, \cdots, k_n)$ be defined and of class C^1 for $k_1 \geqq k_2 \geqq \cdots \geqq k_n$; moreover, assume $\partial\phi/\partial k_i > 0$, $i = 1, \cdots, n$. In an $(n+1)$-dimensional space with constant curvature consider a closed hypersurface S without self-intersection and with bounded principal curvatures $k_1 \geqq k_2 \geqq \cdots \geqq k_n$. If $\phi(k_1, \cdots, k_n)$ has the same sign everywhere on S, then S is a sphere. *H. Busemann* (Cambridge, Mass.)

910:

Thom, René. Un lemme sur les applications différentiables. Bol. Soc. Mat. Mexicana (2) **1** (1956), 59–71.

The author establishes a general approximation lemma for differentiable maps $f: R^n \to R^m$, which embraces and generalizes earlier approximation theorems of Morse, Whitney, and the author. The method is essentially the same as the one used in these earlier works, and consists of imbedding f in a sufficiently rich family of maps. However due to the great generality of the lemma the details are quite difficult to follow.

Let $L(m, n; q)$ be the space of C^q maps of R^n into R^m, in the uniform topology on f and all its partial derivatives of order $\leqq q$. For $f \in L(m, n; q)$ and $r < q$, let F_r (the rth derived of f) be the map which assigns to $x \in R^n$ the values of f and all its partial derivatives of order $\leqq r$ at x. Thus F_r can, after a suitable identification, be interpreted as a map of R^n into a Euclidean space of high dimension, say R^N. Let P be a smooth submanifold of R^N, of codimension p. The lemma now states that if K is a compact subset of R^n and $f \in L(n, m; q)$, then f can be arbitrarily well approximated by maps $g \in L(n, m; q)$, so that the restriction of G_r to K is transversal to P, provided only that $q - r > n - p$.

Here G_r is of course the rth derived of g. [Transversality is taken here in the sense of R. Thom, Ann. Inst. Fourier, Grenoble **6** (1955/56), 43–87; MR **19**, 310.]

Various applications of this lemma are presented in the rest of the paper. The author calls a map $f: R^n \to R^m$ correct if it "encounters all its singularities transversally". (For an understanding of this notion the reader is again

referred to the already cited previous paper of the author.) When $n \geq m$ the author gives the following criterion for a correct map: If $f: R^n \to R^m$ is correct, then the inverse image of a submanifold $P \subset R^m$ either has the same dimension or the same codimension as P.

The author closes with a general discussion of global problems and several examples. He also sketches a proof that a complete real variety has a fundamental cycle mod 2. {We hope that the author will follow up these interesting discussions with a more leisurely and precise account.} *R. Bott* (Ann Arbor, Mich.)

911:

Sumitomo, Takeshi. On the reduction of Minkowski space. J. Fac. Sci. Hokkaido Univ. Ser. I **14** (1958), 50–58.

A Minkowski space is an n-dimensional vector space with a function $F(X)$ which is positively homogeneous of degree one attached to it. A Finsler space is an n-dimensional differentiable manifold to every point of which is attached an n-dimensional Minkowski space with metric function $F(n, X) \equiv F_n(X)$ as a tangent space. The indicatrix at any point x of the Finsler space is a hypersurface of the tangent Minkowski space. If this hypersurface is a hyperquadric then the Minkowski space is a Euclidean space, and the Finsler space is Riemannian. The author is concerned with various conditions under which the Minkowski space reduces to Euclidean space, and this is done by considering the properties of the indicatrix, and in particular of the Riemannian metric defined on the indicatrix. *E. T. Davies* (Southampton)

912:

Watanabe, Shôji. On special Kawaguchi spaces. II. A generalization of affine spaces. Tensor (N.S.) **8** (1958), 169–176.

[For part I, see Tensor (N.S.) **7** (1957), 130–136; MR **19**, 981.] The author considers a special three-dimensional Kawaguchi space whose metric is given by

$$ds = \{A_i(x, x', x'')x''^i + B(x, x', x'')\}^{1/p}dt$$

where certain derivatives of A_i are zero and where Zermelo's conditions for the invariance of arc length are satisfied. This space is called a "generalized affine space".

In the manifold $K_3^{(1)}$ of line elements (x, x') of the above space a "base connection" and, in terms of this, a second generalized affine metric are introduced. Covariant differentiation and curvature tensors are defined. The main theorem is that the vanishing of the curvature tensors is necessary and sufficient that the generalized affine space be a locally affine space.

C. B. Allendoerfer (Seattle, Wash.)

913:

Busemann, Herbert. Spaces with finite groups of motions. J. Math. Pures Appl. (9) **37** (1958), 365–373.

Using results and methods of his book *The geometry of geodesics* [Academic Press, New York, 1955; MR **17**, 779], the author gives simple proofs for I: a two-dimensional compact G-space with negative Euler characteristic has a finite (total) group of motions; II: a compact topological manifold which has non-vanishing Euler characteristic and is metrized as a G-space without conjugate points has a finite group of motions. A more extended, but no deeper, argument proves III: if, in a compact G-space (of dimension greater than 1) without conjugate points, a given class of freely homotopic curves contains at most one closed geodesic, then the group of motions is finite. Riemannian manifolds with negative curvature satisfy the hypotheses of III. In the course of proving III, the author settles some problems raised in his book [loc. cit., appendix]: (17) is answered affirmatively; a counter-example to (19) is given.

L. W. Green (Minneapolis, Minn.)

PROBABILITY

See also 663, 664, 744, 941, 956, 958, 1129, 1223, 1241, 1243, 1244.

914:

Eisenberg, Edmund; and Gale, David. Consensus of subjective probabilities: the pari-mutuel method. Ann. Math. Statist. **30** (1959), 165–168.

The pari-mutuel method of setting odds at horse races is employed as a mechanism for amalgamating individual (subjective) probability distributions to get a group probability distribution. Let p_{ij} be i's subjective probability for alternative (horse) j; let b_i be i's "influence" (the proportion of i's bet to the total stake). If i places a weight (bet) of β_{ij} on j, such that $\sum_j \beta_{ij} = b_i$, the group probability (the pari-mutuel method) of j is $\Pi_j = \sum_i \beta_{ij}$. The group probability distribution given by the Π_j's is in equilibrium with the (β_{ij}) matrix if each i, knowing that odds will be quoted to him in terms of these fixed Π_j's, will not wish to alter his β_{ij}'s. The authors show by a variational method the existence of a unique set of Π_j's for which there can be paired a (β_{ij}) matrix to give an equilibrium state. This resulting set of Π_j's is taken as the group consensus. *H. Raiffa* (Cambridge, Mass.)

915:

Kiesow, Horst. Die Anwendung eines Einfachheitsprinzips auf die Wahrscheinlichkeitstheorie. Arch. Math. Logik Grundlagenforsch. **4** (1958), 27–41.

This paper is concerned with a problem outlined by Carnap [*Logical foundations of probability*, Univ. of Chicago Press, Chicago, Ill., 1950; MR **12**, 664], namely, that of assigning to a given hypothesis a probability relative to the evidence. The author determines such probabilities by certain natural axioms together with a principle called the simplicity principle. This method of assigning probabilities is applicable to hypotheses and evidence statable in the languages defined by Carnap and it is hoped that it will also be applicable to richer languages capable of stating all scientific hypotheses.

A. H. Copeland, Sr. (Ann Arbor, Mich.)

916:

Caregradskiĭ, I. P. On uniform approximation to a binomial distribution by infinitely divisible laws. Teor. Veroyatnost. i Primenen. **3** (1958), 470–474. (Russian. English summary)

It was shown by A. N. Kolmogorov [Teor. Veroyatnost. i Primenen. **1** (1956), 426–435; MR **19**, 586] that if $F^n(x)$ is the cumulative distribution function (c.d.f.) of a sum of

n independent and identically distributed random variables ξ_i $(i=1, 2, \cdots, n)$ then there is an infinitely divisible distribution whose c.d.f. $G(x)$ satisfies

$$(*) \qquad |F^n(x)-G(x)| < Cn^{-1/3} \quad (-\infty < x < \infty),$$

where C is an absolute constant. The author proves that in the special case where each ξ_i takes only the values 1 and 0 with probabilities p and $1-p$ respectively (so that $F^n(x)$ is a binomial c.d.f.) the right-hand side of (*) can be replaced by $C_0 n^{-1/2}$, where C_0 is an absolute constant less than $12\sqrt{7}$. *H. P. Mulholland* (Exeter)

917:

Bergström, Harald. On the limit theorems for convolutions of distribution functions. II. An analysis in the Weierstrass norm. J. Reine Angew. Math. **199** (1958), 1–22.

In this paper the author completes the derivation of results outlined in the first part [see same J. **198** (1957), 121–142; MR **19**, 1088]. Consider sequences of stochastic variables $X_1(n), X_2(n), \cdots, n=1, 2, \cdots$, with the variables independent within each sequence. Form the sums $X_{\lambda k}(n)=X_{\lambda+1}(n)+\cdots+X_{k n}(n)$ with the d.f. $F_{\lambda k}(n, x)$. One of the main results is the following: In order that $F_{0k}(n, x)$ tend weakly to a d.f. $F^{(1)}(x)$ and $F_{k k_n}(n, x)$ tend weakly to a d.f. $F^{(2)}(x)$ as first n and then k tend to ∞, it is necessary and sufficient that $F_1(n, x)-\varepsilon(x)+\cdots+F_{k_n}(n, x)-\varepsilon(x)$ be Cauchy convergent in the Weierstrass norm. Another version of this is given in terms of the normalized d.f. $F_k(n, x+m_k(n))$ where $m_k(n)$ is a truncated expected value of $X_k(n)$. *U. Grenander* (Stockholm)

918:

Reuter, G. E. H. Denumerable Markov processes and the associated contraction semigroups on l. Acta Math. **97** (1957), 1–46.

A "process" is a matrix $(p_{ij})_{i,j\in E}$ (E being a denumerable set) of continuous functions from the non-negative reals to the non-negative reals, satisfying

$$\sum_{j\in E} p_{ij}(t) \leqq 1, \quad p_{ik}(s+t)=\sum_j p_{ij}(s)p_{jk}(t), \quad p_{ij}(0)=\delta_{ij}.$$

$d_i(t)$ is then defined as $1-\sum_j p_{ij}(t)$. If $d_i=0$, all i, the process is called "honest". It is known that

$$\lim_{t\to 0+} t^{-1}(p_{ij}(t)-\delta_{ij})$$

exists, provided $\pm\infty$ are permitted as limits, and

$$(1) \qquad 0 \leqq p_{ij}'(0) < \infty \quad \text{for} \quad i \neq j,$$

$$(2) \qquad \sum_{i\neq j} p_{ij}'(0) \leqq p_{ii}'(0) < \infty.$$

Now let (q_{ij}) be a fixed matrix of real numbers satisfying (1), (2). The operator Q is defined by sending $(x_1, x_2, \cdots)$ to $(y_1, y_2, \cdots)$ with $y_j=\sum_i x_i q_{ij}$, provided both $\sum_j |x_j|$, $\sum_j |y_j| < \infty$. Q_0 is the restriction of Q to finite sequences. A Q-process is a process $(p_{ij}(t))$ such that $p_{ij}'(0)=q_{ij}$, all $i, j\in E$. The "backward" and "forward" equations of Kolmogorov say, respectively, (B_{ij}) $p_{ij}'(t)$ exists, is continuous and equals $\sum_k q_{ik}p_{kj}(t)$, (F_{ij}) $p_{ij}'(t)$ exists, is continuous, and equals $\sum_k p_{ik}(t)q_{kj}$. It actually is equivalent to assume that the derivatives only exist almost everywhere, as is shown.

Further, (B_{ij}) or (F_{ij}) implies $p_{ij}'(0)=q_{ij}$. Conditions are given for existence and uniqueness of processes satisfying various combinations of (B_{ij}), $F_{ij})$, and "honesty".

Theorem 2: Given Q, a process satisfies (B_{ij}) for all i, $j\in E \leftrightarrows$ its infinitesimal generator Ω is an extension of Q_0. Theorem 3: For fixed $i\in E$, and a Q-process (p_{ij}), (B_{ij}) holds for all $j\in E \leftrightarrows \sum_j q_{ij}+d_i'(0)=0$.

Theorem 4: (F_{ij}) holds for all $i, j\in E \leftrightarrows \Omega$ is a restriction of Q. Feller has constructed an example of a Q-process $\mathscr{T}=(F_{ij})$, for given Q, which satisfies both the forward and backward equations. Q is called conservative if $\sum_j q_{ij}=0$, all i.

Theorem 6: For conservative Q, if (F_{ij}) is honest then it is the only Q-process, while otherwise there are even infinitely many honest Q processes.

Theorem 8: Given Q, the following conditions are equivalent, λ being any number >0: (a) The system $(\lambda+q_i)\xi_i=\sum_{j\neq i} q_{ij}\xi_j$ has no nonzero bounded solution. (b) The above system has no nonzero non-negative bounded solution. (c) There is only one Q-process. If these fail, then there are infinitely many Q-processes.

Let n^+ be the maximal number of linearly independent non-negative elements of the nullspace of $\lambda I-Q$ (this is shown independent of λ).

Theorem 10: (i) If $\mathscr{T}$ is honest, or if $\mathscr{T}$ is not honest and $n^+=0$, then there is exactly one process with $\Omega\subset Q_0$. (ii) If $\mathscr{T}$ is not honest, and $n^+=1$, there are infinitely many processes with $\Omega\subset Q$, but only one honest one. (iii) If $\mathscr{T}$ is dishonest and $n^+>1$, then there are ∞ many honest processes with $\Omega\subset Q$.

Examples are given illustrating various combinations of uniqueness and non-uniqueness in the solutions to the backward and forward equations.

J. Feldman (Berkeley, Calif.)

919:

Reuter, G. E. H. Denumerable Markov processes. II. J. London Math. Soc. **34** (1959), 81–91.

Notation as in above review. Given Q, this paper gives a method for constructing Q-processes which yields substantially all the examples of the above paper, and which in a certain class of cases gives all Q-processes (except the Feller process).

J. Feldman (Berkeley, Calif.)

920a:

Urbanik, K. The values at the fixed moment of generalized stochastic processes. Acta Math. Sinica **8** (1958), 146–152. (Chinese. English summary)

920b:

Urbanik, K. The values at the fixed moment of generalized stochastic processes. Sci. Sinica **7** (1958), 1–9. [These are two versions of the same article.]

A sequence $\phi_s(\omega, t)$ of generalized stochastic processes (gstp.) [see K. Urbanik, Teor. Veroyatnost. i Primenen. **1** (1956), 146–149; Studia Math. **16** (1958), 268–334; MR **19**, 326; **20** #4309] converges in probability to a gstp. $\phi(\omega, t)$ when $s\to 0$ if there exist an integer k and continuous stochastic processes $F_s(\omega, t)$, $F(\omega, t)$ such that

$$\frac{d^k}{dt^k} F_s(\omega, t) = \phi_s(\omega, t), \quad \frac{d^k}{dt^k} F(\omega, t) = \phi(\omega, t),$$

and F_s converges in probability to F uniformly in each finite t-interval. $\phi(\omega, t)$ has the value $\phi(\omega, t_0)$ at t_0 if $\phi(\omega, et+t_0)$ converges in probability to $\phi(\omega, t_0)$, when

$\varepsilon \to 0$. It follows almost immediately [compare Z. Zieleźny, Bull. Acad. Polon. Sci. Cl. III **3** (1955), 519–520; MR **17**, 595] that this value is a random variable and if $\phi(\omega, t)$ is a strictly stationary gstp. with independent values, then this random variable is a constant almost surely. Let $f(\omega, t)$ be a homogeneous stochastic process with independent increments. $d^s f(\omega, t)/dt^s$ $(s \geqq 1)$ has a value at t_0 if and only if $\lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} (\varepsilon^{2s} + u^2)^{-1} dG(u)$ exists $(s = 1)$ and is equal to zero $(s \geqq 2)$, where $G(u)$ is defined according to the Lévy-Hinčin formula for the infinitely divisible law of $f(\omega, t+\delta) - f(\omega, \delta)$.

L. Schmetterer (Berkeley, Calif.)

921:

Wang, Shou-jen. A note on generalized stochastic processes. Sci. Record (N.S.) **2** (1958), 15–18.

If $\phi(\omega, t)$ is a gstp. [compare the review above] and if for each $\varepsilon > 0$ there is a representation $f_{n,\varepsilon}(\omega, t)$ of $\phi(\omega, t)$ such that $f_{n,\varepsilon}(\omega, t)$ is a continuous process with ε-independent increments for each n, then $\phi(\omega, t)$ is a gstp. with independent increments. This is the solution of a problem of A. Prékopa. A similar theorem holds for gstp. with independent values. *L. Schmetterer* (Berkeley, Calif.)

922:

Chung, K. L. Some aspects of continuous parameter Markov chains. Publ. Inst. Statist. Univ. Paris **6** (1957), 271–287.

Let $(p_{ij}(t))$, $i, j = 0, 1, \cdots$, $t > 0$, be a semigroup of Markov transition matrices, with $\sum_j p_{ij}(t) = 1$. Assume $\lim_{t \downarrow 0} p_{ii}(t) = 1$ and define $p_{ij}(0) = \delta_{ij}$. The p_{ij} correspond to a suitably standardized Markov process $x(t, w)$. In general, some or all of the states may be instantaneous, and the fictitious state ∞ may be adjoined. Let

$$S_i(w) = \{t : x(t, w) = i\},$$

and let m denote Lebesgue measure. Some of the results are as follows.

$$P(S_i \text{ is unbounded} | x_0 = i) = P(m(S_i) = \infty | x_0 = i);$$

the common value is 0 if $\int_0^\infty p_{ii}(t)dt < \infty$ and 1 if $\int_0^\infty p_{ii}(t)dt = \infty$; in the latter case i is called recurrent. If H is a set of states (the "taboo set") and $i \notin H$, define

$$_H p_{ij}(t) = P\{x_s \notin H,\ 0 < s < t;\ x_t = j | x_0 = i\};$$

the definition is appropriately modified if $i \in H$. Then $_H p_{ij}$ is continuous in $[0, \infty)$, and $\int_0^\infty {}_H p_{ij}(t)dt < \infty$ provided $\sum_{k \in H} p_{jk}(t) > 0$ for some t. The notation $i \to j$ means that $p_{ij}(t) > 0$ for some t. If $i \to j$ and $j \to i$, and $j \neq i$, then

$$0 < \lim_{t \to \infty} \int_0^t p_{ii}(x)dx \Big/ \int_0^t p_{jj}(x)dx = \int_0^\infty {}_j p_{ii}(x)dx \Big/ \int_0^\infty {}_i p_{jj}(x)dx < \infty.$$

This relation holds even if i and j are instantaneous states; two other expressions for the limit are given that are meaningful only if i and j are stable. {On the bottom half of p. 281, lower case p should be replaced by capital P in several places; on the same page, the author has informed the reviewer that the fact that $_i p_{ij}(t) \to 0$ follows from the fact that the matrix $(_i p_{kj})$, for fixed i and variable k and j, is substochastic, rather than from continuity.} Some of the results depend on an earlier paper of the author [Proc. Third Berkeley Symposium on Math. Statist. and Probability, vol. II, 29–40, Univ. of California Press, Berkeley, 1956; MR **18**, 940].

T. E. Harris (Santa Monica, Calif.)

923:

Karlin, Samuel; and McGregor, James. A characterization of birth and death processes. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 375–379.

A regular Markov matrix $P(t) = (P_{ij}(t))$ satisfies the conditions: $P_{ij} \geqq 0$, $\sum_j P_{ij} \leqq 1$, $P(t+s) = P(t)P(s)$, $\lim P_{ij}(t) = \delta_{ij}$, $\lim (1 - P_{ii}(t))/t = q_i$ $(0 < q_i < \infty)$, and $\lim P_{ij}(t)/t = a_{ij}$ exists when $i \neq j$ and is finite (all limits as $t \to 0+$). A matrix $A = (a_{ij})$ is strict Jacobi if $a_{ij} > 0$ when $|i-j| = 1$; $a_{ii} < 0$; $a_{ij} = 0$ when $|i-j| > 1$; $\sum_j a_{ij} \leqq 0$. A regular Markov matrix is strictly totally positive [strictly positive of order n] if all minors [of order $\leqq n$] are strictly positive for every $t > 0$. An integer-valued path function $x(t)$ is called "continuous" if it is continuous to the right in the ordinary sense and the only discontinuities are simple discontinuities with saltus ± 1, with an appropriate modification for continuity at ∞. Theorem. For a regular Markov matrix P the following conditions are equivalent: (1) There is a realization with continuous path functions which has the strong Markov property and whose states form a single communicating class; (2) P is strictly totally positive; (3) P is strictly positive of order 2; (4) $A = P'(0)$ is a strict Jacobi matrix and $P'(t) = AP(t) = P(t)A$. That (1)$\to$(2) is proved in a paper by the authors now in press. (2)$\to$(3) is obvious. As for (3)$\to$(4): the authors have shown elsewhere [Trans. Amer. Math. Soc. **85** (1957), 489–546; MR **19**, 989] that (3) implies that $P'(0)$ is Jacobi; they now show the strictness; (3) also implies that $P_{kj}(s)/s$ is bounded uniformly in k and s, $k > j+1$, $0 < s < \varepsilon$, when j is fixed, which leads to the forward equation $P' = PA$. To prove (4)$\to$(1), use is made of results of Chung [Proc. Third Berkeley Symposium on Math. Statistics and Probability, 1954–1955, vol. II, pp. 29–40, Univ. of California Press, Berkeley-Los Angeles, 1956; MR **18**, 940] and it is shown that, roughly speaking, if $i < j < k$, or $i > j > k$ then the first passage time from i to k is the time from i to j plus the time from j to k. *T. E. Harris* (Santa Monica, Calif.)

924:

★de Finetti, Bruno. Sul numero di elementi al di là dell'ultimo osservato. Scritti matematici in onore di Filippo Sibirani, pp. 95–107. Cesare Zuffi, Bologna, 1957.

C'est une variante du processus "birth and death", expliquée par l'exemple suivant. Un observateur contrôle tous les nombres de plaque des automobiles qu'il rencontre (se bornant aux seuls nombres progressifs d'un département donné), et il enregistre toujours le maximum observé; il se demande quel est le maximum existant, c'est à dire le nombre des éléments au delà du maximum observé.

En supposant qu'entre l'instant t et l'instant $t+dt$ un nouveau élément est ajouté avec probabilité λdt (λ = const.) et qu'un élément donné y est observé avec probabilité μdt (μ = const.) on obtient un processus stochastique $X(t)$, où $X(t)$ peut prendre les valeurs 0, 1, 2, $\cdots$, peut s'accroître seulement par des sauts unitaires, et décroître par des sauts quelconques. Les probabilités $p_n(t) = P[X(t) = n]$ et $P_n(t) = P[X(t) \geqq n]$ sont calculées en fonction des valeurs initiales $p_n(0)$ et $P_n(0)$ et du paramètre

$a=\lambda/\mu$ (où l'on peut supposer avec un convenable choix de l'unité de temps $\mu=1$, c'est à dire $a=\lambda$). Le cas stationnaire ($p_n=\lim_{t\to\infty} p_n(t)$; $P_n=\lim_{t\to\infty} P_n(t)$) est traité en détail; valeurs numériques de l'espérance mathématique $m(a)$ y sont données soit se servant de son expression exacte, pour petites valeurs entières du paramètre a, soit employant des formules d'approximation dues à Ramanujan. On étudie ensuite la distribution des sauts, soit absolue, soit conditionnée à des hypothèses données, et la probabilité que $X=n$ après un saut (déterminé ou non), ou après un période sans sauts négatifs. Une estimation soit de a, soit de $m(a)$ est obtenue en admettant que la moyenne arithmétique de sauts observés, $\bar{s}$, soit une bonne approximation de son espérance mathématique: on trouve $a\simeq\frac{1}{2}\pi\bar{s}^2$, $m(a)\simeq\frac{1}{2}\pi\bar{s}$.

D. Fürst (Rome)

925:

Whittle, P. On the use of the normal approximation in the treatment of stochastic processes. J. Roy. Statist. Soc. Ser. B. **19** (1957), 268–281.

In this paper, Whittle develops an approximation method for treating nonlinear Markov processes by neglecting third- and higher-order cumulants, i.e. by assuming the variate is normal. Let x be the vector variate of a Markov process with characteristic function $\phi(\theta)=E(e^{\theta x})$. Whittle shows that Bartlett's relation $\partial\phi(\theta)/\partial t=L(\theta,\ \partial/\partial\theta)\phi(\theta)$, where $L(\theta,x)$ is the derivate cumulant function, can be written

$$\frac{\partial\psi(\theta)}{\partial t}=[e^{G(\theta,\partial/\partial x)}L(\theta,x)]_{x=\psi_1(\theta)}$$

where $\psi(\theta)=\log\phi(\theta)$ and $\psi(\theta+\xi)-\psi(\theta)=\xi'\psi_1(\theta)+G(\theta,\ \xi)$ (the prime indicates the transpose). If $L(\theta,x)$ can be expanded as $L(\theta,x)=\theta'F_1(x)+\frac{1}{2}\theta'F_2(x)\theta+\cdots$, then the first and second order terms of Whittle's form of Bartlett's relation yield, respectively,

$$\frac{\partial m}{\partial t}=\Omega F_1(m),$$

$$\frac{\partial V}{\partial t}=\Omega\left[\frac{\partial F_1(m)}{\partial m}V+\left(\frac{\partial F_1(m)V}{\partial m}\right)'+F_2(m)\right],$$

where $\Omega=\exp[\frac{1}{2}(\partial/\partial m)'V(\partial/\partial m)]$. From these equations, m and V for the normal approximation $\psi(\theta)=\theta'm+\frac{1}{2}\theta'V\theta$ of the process can be determined. This normalization method is applied to models of logistic population growth, plant competition, and animal predation. The Bartlett relation is generalized to provide equations for the characteristic functional of the process. Whittle's results indicate that: (1) the normalization method is best suited to the case of "strong equilibrium" (i.e. the variate is bound to an equilibrium point so strongly that the restoring force rarely becomes non-linear) and "continuity" (i.e. the size of the interval over which the variate wanders freely is large compared with the change in variate value at any one transition); and (2) the normalization method in the stochastic model is a more adaptable technique than is the linearization method in the corresponding deterministic model, in that nonlinear effects can be demonstrated more faithfully and less stringent conditions need be fulfilled for asymptotic approximations.

E. A. Robinson (Madison, Wis.)

926:

Čistyakov, V. P. Generalization of a theorem for branching processes. Teor. Veroyatnost. i Primenen. **4** (1959), 109–113. (Russian. English summary)

The author considers a temporally homogeneous branching process with n types $T_1, \cdots, T_n$ and a continuous time parameter t. Let e^{At} be the matrix whose (i,j)th element is the expected number of objects of type j at time t if originally there is one object of type i. Let λ be the characteristic root of a of maximum real part (it is known from the theory of Frobenius that λ is real). It was shown by Jiřina [Czechoslovak Math. J. **7**(**82**) (1957), 130–153; MR **20** #1361] that if $\lambda<0$ and a is indecomposable, then there is a conditional limiting distribution for the numbers of objects of different types as $t\to\infty$, given that the total number of objects is not 0. The author obtains a result valid when a is decomposable. Let $Q_i(t)$ be the probability that some particles still exist at t if initially there is one particle of type i. If $\lambda<0$, then there is a non-negative integer k such that

$$\lim_{t\to\infty} Q_i(t)/t^k e^{\lambda t}=K_i\quad(0<K_i<\infty),$$

the conditional probabilities have limits, and the generating function for the conditional probabilities satisfies a functional equation similar to that satisfied in the indecomposable case. The exact determination of the integer k depends on a classification of the relations between types.

T. E. Harris (Santa Monica, Calif.)

927:

Gladyšev, E. G. On multi-dimensional stationary random processes. Teor. Veroyatnost. i Primenen. **3** (1958), 458–462. (Russian. English summary)

Let

$$x(t)=(x_1(t),\ x_2(t),\ \cdots,\ x_n(t))$$

be an n-dimensional stationary (in the wide sense) process. The author generalizes to the n-dimensional case the Wold decomposition theorem for ordinary stationary processes and the necessary and sufficient conditions for regularity of a stationary process given by A. Kolmogorov [Izv. Acad. Nauk SSSR. Ser. Mat. **5** (1941), 3–14; MR **3**, 4].

A discussion of the corresponding results for discrete time multi-dimensional stationary processes was given by Wiener and Masani [Acta Math. **98** (1957), 111–150; MR **20** #4323].

J. L. Snell (Hanover, N.H.)

928:

★Henze, Ernst. Beiträge zum Irrfahrtproblem. Thesis, Technische Hochschule Stuttgart, 1958. 155 pp.

A point performs random walks in a square grid in the plane with transition probabilities p_1, q_1, p_2, q_2 ($p_1+q_1+p_2+q_2=1$) to the four nearest grid neighbors. Three regions are considered: The whole plane, the upper half plane, and the plane with the negative real axis removed. Various questions connected with random walks in these regions are investigated, such as the probability of reaching a given point in n steps, the expected number of visits at a point, the probability of absorption at the boundary, mean duration and generating function. The aim of the author is the construction of explicit solutions of the difference equations that characterize the quantities in question. For the special case $p_1=q_1=p_2=q_2=\frac{1}{4}$, to which the first part of the paper is devoted, many of the

expressions obtained are equal or equivalent to results obtained earlier by W. H. McCrea and F. J. W. Whipple [Proc. Roy. Soc. Edinburgh **10** (1940), 281–298; MR **2**, 107], A. Stöhr [Math. Nachr. **3** (1950), 208–242, 295–315, 330–357; MR **12**, 711] and other authors. The formulas for the general case appear to be new. For many of these formulas the author derives asymptotic expressions for vanishing mesh length. *W. Wasow* (Madison, Wis.)

929:

Takács, Lajos. Remarks on random walk problems. Magyar Tud. Akad. Mat. Kutató Int. Közl. **2** (1957), 3/4, 175–182. (Hungarian and Russian summaries)

The author gives simple proofs for some known limit theorems for random walks with or without two absorbing barriers. *E. Lukacs* (Washington, D.C.)

930:

Mercer, A.; and Smith, C. S. A random walk in which the steps occur randomly in time. Biometrika **46** (1959), no. 1/2, 30–35.

"The steps of a one-dimensional random walk are positive and occur randomly in time at a fixed mean rate. The sizes of the steps are independent and the size of each step has the same given probability distribution. The distribution of the time to reach a fixed barrier is obtained and approximations to its moments are derived. The results are extended to the case in which the barrier and the random walk process converge at a constant rate." (From the authors' summary.)

M. Muller (New York, N.Y.)

931:

Bažarin, G. P. Final probabilities for multi-dimensional Markov processes which describe the action of some two-stage telephone busy-signal systems. Teor. Veroyatnost. i Primenen. **3** (1958), 452–458. (Russian. English summary)

The author analyzes a two stage telephone system. The first stage consists of switches A_i ($i=1, 2, \cdots, k$), where A_i has n_i incoming and m_i outgoing lines and is subject to Poisson-type calls occurring at a rate λ_i with exponentially distributed durations of mean length μ_i. The $\sum_1^k m_i$ outgoing lines from $\bigcup_1^k A_i$ form the incoming lines to a switch B, which, in turn, has l outgoing lines. A matrix-valued stochastic process $\xi_{ij}(t)$ is defined as $\xi_{ij}(t)=1$ if a call on A_i occupies outgoing line j from B, $\xi_{ij}(t)=0$ otherwise. Formulas, too complicated to reproduce here, are found for $\lim_{t\to\infty} \Pr\{\xi_{ij}(t)=s_{ij}\}$ for the case when (1) the outgoing line from B is chosen at random from among the free lines, and (2) there are preassigned probabilities η_{ij} that a call placed on A_i will be addressed to the jth outgoing line of B. The author also considers distributions of the number of busy lines. *E. Reich* (Minneapolis, Minn.)

932:

Bennett, W. R. Statistics of regenerative digital transmission. Bell System Tech. J. **37** (1958), 1501–1542.

The paper is concerned with stochastic processes of the type $x(t)=\sum_{n=-\infty}^{\infty} a_n g(t-nT)$, where $g(t)$ is a non-random "pulse", and the a_n's are an ergodic sequence of 0's and 1's. For various specific $g(t)$ the spectral density of $x(t)$ is computed and graphed. The major question considered is the extent to which a timing wave can be recovered from $x(t)$, possibly in the presence of noise, by the application of operations to $x(t)$. This is of interest when $x(t)$ represents a message transmitted by means of pulse code modulation. *E. Reich* (Minneapolis, Minn.)

933:

Kay I.; and Silverman, R. A. Multiple scattering by a random stack of dielectric slabs. Nuovo Cimento (10) **9** (1958), supplemento, 626–645.

This paper deals with a statistical one-dimensional scattering problem for which the contributions produced by a different number (n) of successive scatterings are compared with each other numerically. The problem concerns a finite number (N) of parallel infinite plane slabs (of thickness d), the dielectric constants of which differ by an amount $\beta\Delta\varepsilon$ or $-\beta\Delta\varepsilon$ from the value 1 holding outside the stack of slabs. The wave function $u(z)$ generated by a plane wave $\exp(ikz)$ (impinging perpendicularly to the slabs) then satisfies the integral equation:

$$(1)\quad u(z) = \exp(ikz) - \frac{k\Delta\varepsilon}{2i}\int_0^{Nd} \exp(ik|z-z'|)\,\beta(z')u(z')dz';$$

$\beta(z)$ equals either $+1$ or -1 in each individual slab $(r-1)d<z<rd$ $(1\le r\le N)$. The nth term of the Neumann expansion of the solution of (1) represents the sum of all contributions that have suffered n consecutive scatterings at the slab boundaries. A sufficient condition for the convergence of this expansion reads: (2) $\frac{1}{2}kd\Delta\varepsilon N<1$.

The transmission and reflection coefficients T and R of the whole stack, too, can be expanded into contributions due to a special number n of individual scatterings. The author discusses the averages $\langle|T|^2\rangle$ and $\langle|R|^2\rangle$ that are associated with a random distribution of the values $+1$ and -1 for the quantity β fixing the dielectric constant inside the N individual slabs. The $(2n)$th term of the expansion for $\langle|T|^2\rangle$ proves to have an upper bound $(2n+1)(kd\Delta\varepsilon/2)^{2n}\langle\beta_N{}^{2n}\rangle$; in this expression $\langle\beta_N{}^{2n}\rangle = 2^{-N}\sum_{r=0}^{N}\binom{N}{r}(N-2r)^{2n}$ represents the average of the $(2n)$th power of the sum β_N of N independent variables taking on the values $+1$ and -1 with equal probability. For $2n\ll N$ the latter quantity can be approximated by $2^1(2nN/e)^n$, in connection with a nearly Gaussian distribution of β_N in this domain; the other approximation $\langle\beta_N{}^{2n}\rangle\sim 2^{1-N}N^{2n}$ holds for $2n\gg N$. A numerical discussion of the complete $(2n/N)$ range demonstrates that the first terms of the expansion for both $|T|^2$ and $|R|^2$ may fall off rapidly with n, even if the limit of the convergence condition (2) is approached, and in spite of a possible maximum contribution for some n value in the domain $2n>N$. The result obtained here, that a large number N of scattering elements generally involves rapidly decreasing contributions for low n values, may be important in view of a corresponding property for the scattering of radio waves in three-dimensional inhomogeneous media.

H. Bremmer (Eindhoven)

934:

Scheffler, H. Strahlenoptische Ausbreitung in Medien mit statistisch verteilten Inhomogenitäten. II. Streuung in kleine Winkelbereiche. Astr. Nachr. **284**, 269–274 (1959).

Continuing the work of a previous paper [same Nachr. **284**, 227–232 (1958); MR **20** #6165], the author calculates the 4-variate probability density of the angular and lateral

deflections of a ray penetrating a medium with statistically homogeneous refractive index fluctuations. He then considers the more general case of a medium which is not statistically homogeneous; in particular, the case where the mean and variance of the refractive index vary spatially. By making suitable simplifying assumptions, explicit expressions are obtained for the parameters of the above-mentioned joint probability density.

R. A. Silverman (New York, N.Y.)

STATISTICS

See also 914, 925, 933, 1223, 1238, 1242.

935:

Fisher, R. A. Mathematical probability in the natural sciences. Metrika **2** (1959), 1–10.

Expository paper restating the views of the author on the so-called fiducial argument and fiducial probability.

J. Neyman (Berkeley, Calif.)

936:

Quenouille, M. H. Tables of random observations from standard distributions. Biometrika **46** (1959), no. 1/2, 178–202.

A table of 1,000 pseudo observations from eight distributions is given. The distributions have zero means and unit variances. They are the normal, a rectangular, a log normal, an exponential, three distributions derived from an Edgeworth type A expansion (using Cornish-Fisher approximations), and a two-sided exponential. The listed observations were obtained by: (1) taking 1,000 pseudo random observations which represent the first two pages of H. Wold's *Random normal deviates* [Cambridge Univ. Press, 1948; MR **10**, 553]; (2) selecting the associated 1,000 percentage points of the normal distribution; and (3) transforming these percentage points to give the associated values of "observations" from the seven distributions mentioned above.

M. Muller (New York, N.Y.)

937:

Pearson, E. S. Note on Mr. Quenouille's Edgeworth Type A transformation. Biometrika **46** (1959), no. 1/2, 203–204.

Three of the distributions used by Quenouille [see previous review] are obtained from transformations of a normally distributed variable via an Edgeworth type A expansion through the Cornish-Fisher equations [loc. cit., p. 179]. The purpose of this paper is to compare the differences in the frequency functions resulting from using the Cornish-Fisher equations instead of the Edgeworth type A expansions given by equation (1). The results imply that the differences are not significant.

M. Muller (New York, N.Y.)

938:

★Castellano, Vittorio. Sulla suddivisione in intervalli di una variabile già divisa in intervalli nella ipotesi che la distribuzione delle frequenze sia una poligonale. Scritti matematici in onore di Filippo Sibirani, pp. 63–80. Cesare Zuffi, Bologna, 1957.

With reference to an earlier note [Statistica **16** (1956), 151–186; MR **18**, 520], corrections are given for the two first moments of a grouped distribution, under the assumption that the given random variable has a polygonal frequency function. In the case of a general continuous fr. f. the preceding one is considered as an approximation and some properties of the interpolating polygonals are studied. Other corrections are found by subgrouping the grouped variable and supposing the fr. f. linear in each interval.

The author observes that the aim of this kind of research is to investigate the characteristics of the original sample distribution (i.e. of the non-grouped empirical distribution), by means of grouped characteristics, whilst the relationship between the grouped distribution of the sample and the distribution of the population is quite another problem. For this reason, in the author's opinion, Sheppard's corrections are misplaced in a chapter on sampling distributions, as it occurs in Cramér's *Mathematical methods of statistics* [Princeton Univ. Press, Princeton, N. J., 1946; MR **8**, 39].

D. Fürst (Rome)

939:

Kimball, A. W.; and Leach, E. Approximate linearization of the incomplete β-function. Biometrika **46** (1959), no. 1/2, 214–218.

The authors consider two normalising transformations of the incomplete beta distribution of x:

$$I_x(p, q) = \int_0^x t^{p-1}(1-t)^{q-1}dt \Big/ \int_0^1 t^{p-1}(1-t)^{q-1}dt = \alpha,$$

suitable, e.g., for use in fitting to dose-response curves where the dose depends on x, and α is the probability of response. Both formulae are in terms of the χ^2 distribution with $2q$ degrees of freedom, and with the Wilson-Hilferty normalising approximation to χ^2 [Proc. Nat. Acad. Sci. U.S.A. **17** (1931), 684–688]. One of them is "linear" in that

$$y_\alpha = A + BX,$$

where y_α is the unit normal deviate corresponding to α, A is a function of q only, B of p and q, and X is a complex empirical function of x only. In numerical tables some calculated values of α are compared with exact values and with the asymptotic normal approximation to $I_x(p, q)$.

M. E. Wise (Sutton)

940:

Pillai, K. C. Sreedharan; and Samson, Pablo, Jr. On Hotelling's generalization of T^2. Biometrika **46** (1959), no. 1/2, 160–168.

Suppose $(x_{11\alpha}, \cdots, x_{1p\alpha}, \alpha = 1, \cdots, n_1)$ and $(x_{21\beta}, \cdots, x_{2p\beta}, \beta = 1, \cdots, n_2)$, $n_2 > p$, are two samples from p-dimensional normal distributions. Let $(\bar{x}_{11}, \cdots, \bar{x}_{1p})$ and $(\bar{x}_{21}, \cdots, \bar{x}_{2p})$ be the vectors of means of the two samples. Let

$$S_{1ij} = \sum_\alpha (x_{1i\alpha} - \bar{x}_{1i})(x_{1j\alpha} - \bar{x}_{1j}),$$

$$S_{2ij} = \sum_\beta (x_{2i\beta} - \bar{x}_{2i})(x_{2j\beta} - \bar{x}_{2j}),$$

and let S_1, S_2 be the two matrices $\|S_{1ij}\|$ and $\|S_{2ij}\|$, respectively. The quantity T_0^2 defined by $T_0^2/(n_2-1) = \text{trace}\,(S_2^{-1}S_1)$ is, in a certain sense, a generalization of Hotelling's T^2. But trace $(S_2^{-1}S_1)$ is the sum of the roots of the characteristic equation $|S_1 - \lambda S_2| = 0$. If t is the number of non-zero roots of this equation and if we let $U^{(t)} = \text{trace}\,(S_2^{-1}S_1)$ the authors have given explicit

formulas for the first four moments of $U^{(t)}$ for $t=2, 3, 4, 5, 6$ under the assumption that the covariance matrices of the two normal distributions from which the two samples came are identical. The exact distributions of $U^{(1)}$ and $U^{(2)}$ were found by Hotelling in 1931 and 1951, respectively.

Putting $m=\frac{1}{2}(n_1-t-2)$, $n=\frac{1}{2}(n_2-t-2)$, the authors give exact values of the upper 5% and 1% points of the distribution of $U^{(t)}$ for $t=2$ for certain combinations of values of m and n. They give approximations to the upper 5% and 1% points of $U^{(t)}$ for $t=3$, 4 for various combinations of values of m and n.

S. S. Wilks (Princeton, N.J.)

941:

Zinger, A. A. Independence of quasi-polynomial statistics and analytical properties of distributions. Teor. Veroyatnost. i Primenen. **3** (1958), 265–284. (Russian. English summary)

Results previously announced by the author for polynomials [Dokl. Akad. Nauk. SSSR **110** (1956), 319–322; MR **19**, 991] are here stated and proved for "quasi-polynomials". A real-valued function $S(X)$ of the vector X is said to be a quasi-polynomial if there exist a continuous function ϕ and two non-negative polynomials $r(X)$ and $R(X)$ such that $r(X)\leq\phi[S(X)]\leq R(X)$ for all X. Theorem 5 of the author is an improvement on a result due to Linnik [Vestnik Leningrad. Univ. **11** (1956), no. 1, 35–48; MR **17**, 983]. *S. G. Ghurye* (Chicago, Ill.)

942:

Aĭvazyan, S. A. A comparison of the optimal properties of the Neyman-Pearson and the Wald sequential probability ratio tests. Teor. Veroyatnost. i Primenen. **4** (1959), 86–93. (Russian. English summary)

Suppose H_θ is the hypothesis that a random variable x has probability density function $f(x, \theta)$, where θ is an unknown (real vector) parameter. Consider two alternatives H_{θ_0} and H_{θ_1} where θ_1 is near θ_0. Assume that $f(x, \theta)$ satisfies conditions sufficient to permit a Neyman-Pearson likelihood ratio test for testing H_{θ_1} against H_{θ_0}. For given probabilities α and β of type I and type II errors, respectively, let n_1 be the fixed-sample size required for a Neyman-Pearson test of H_{θ_1} against H_{θ_0}. For the same values of α and β and for any value of θ in the neighborhood of θ_0, let $n_2(\theta)$ be the average sample size required for a Wald sequential test for H_{θ_1} against H_{θ_0} given that θ is the true value of the parameter. It is shown under certain regularity conditions that as $\theta_1\to\theta_0$ the limit of $n_2(\theta_1)/n_1$ is a function $\tilde{\omega}(\alpha, \beta)$ depending only on α and β. A table of values of $\tilde{\omega}(\alpha, \beta)$ is given for values of α and β ranging between 0.0001 and 0.7.

S. S. Wilks (Princeton, N.J.)

943:

Pfanzagl, J. Ein kombiniertes Test & Klassifikations-Problem. Metrika **2** (1959), 11–45.

"A decision process is considered which consists of two steps: First, a nullhypothesis H_0 is to be tested. If H_0 is rejected, a decision is to be made as to which of the alternative hypotheses $H_1, H_2, \cdots, H_k$ is valid. This second step is called 'classification'. It is assumed, that in case H_0 is not valid, each of the alternative hypotheses $H_1, H_2, \cdots, H_k$ has the same probability. Starting with this assumption, an optimal decision process is developed which has a specified level of significance α (i.e. by which the nullhypothesis H_0 is accepted with probability α, if it is valid), and for which the probability of a correct classification is a maximum in the case where the nullhypothesis is not valid." (From the author's summary.) There are a number of applications.

M. Dwass (Evanston, Ill.)

944:

Lancaster, H. O. The structure of bivariate distributions. Ann. Math. Statist. **29** (1958), 719–736.

K. Pearson in his study on the association between two chance variables defined a measure, the mean square contingency, $\phi^2=\chi^2/N$, where χ^2 is that usually calculated in a contingency table with fixed marginal totals and N is the size of the sample. As the sample size becomes large and the subdivisions are made finer, ϕ^2 tends to a limiting value which depends on the bivariate distribution. The author obtains a canonical representation for ϕ^2 bounded distributions. This is given by

$$dF(x, y) = \{1+\sum \rho_t x^{(t)}y^{(t)}\}dG(x)dH(y)$$

with $\phi^2=\sum \rho_t^2$, where F is the joint cdf, G and H are the marginal cdf's, $x^{(t)}$ and $y^{(t)}$ are suitable selected canonical variables which form complete sets of orthonormal functions with respect to the marginal distributions. In fact $x^{(t)}$ and $y^{(t)}$ are selected so as to have mean 0 and variance 1 and to maximize their correlation subject to $x^{(t)}$ being uncorrelated with $x^{(1)}, x^{(2)}, \cdots, x^{(t-1)}$ and subject to $y^{(t)}$ being uncorrelated with $y^{(1)}, y^{(2)}, \cdots, y^{(t-1)}$.

H. Chernoff (Stanford, Calif.)

945:

Hooper, John W. The sampling variance of correlation coefficients under assumptions of fixed and mixed variates. Biometrika **45** (1958), 471–477.

Let

$$\{y_{1t}, \cdots, y_{Mt}; x_{1t}, \cdots, x_{\Lambda t}\} \quad (t = 1, \cdots, T)$$

be T vectors of observations on $M+\Lambda$ variables $y_1, \cdots, y_M; x_1, \cdots, x_\Lambda$, and let the $y_{\mu t}$ and $x_{\lambda t}$ ($\mu=1, \cdots, M$; $t=1, \cdots, T$; $\lambda=1, \cdots, \Lambda$) satisfy the relations

$$y_{\mu t} = \sum_{\lambda=1}^{\Lambda} \pi_{\mu\lambda}x_{\lambda t}+u_{\mu t}; \quad x_{\lambda t} = \xi_{\lambda t}+w_{\lambda t},$$

where the $\xi_{\lambda t}$ are non-stochastic real numbers, the $\pi_{\lambda\mu}$ are parameters independent of t, and the T vectors $\{w_{1t}, \cdots, w_{\Lambda t}\}$ are independent observations on a Λ-dimensional normal distribution with zero means and are independent of the T "disturbance" vectors $\{u_{1t}, \cdots, u_{Mt}\}$, which themselves are independent observations on an M-dimensional normal distribution with zero means.

For this general model the author finds expressions for the asymptotic variance of an estimate of the correlation coefficient of x_λ and y_μ ($\lambda=1, \cdots, \Lambda$; $\mu=1, \cdots, M$), and for the asymptotic covariance of two such coefficients. Various special cases are discussed.

D. M. Sandelius (Göteborg)

946:

Pillai, K. C. Sreedharan; and Bantegui, Celia G. On the distribution of the largest of six roots of a matrix in multivariate analysis. Biometrika **46** (1959), no. 1/2, 237–240.

This paper is related to the work of Roy which shows that tests of hypotheses and estimates of confidence

intervals in multivariate situations can be obtained from the distribution of the extreme characteristic roots of a matrix which is a function of the observations. In particular this paper studies the cumulative distribution function of the largest of six roots of this matrix. An approximate method of determining upper percentage points (5% or less) to the C.D.F. of the largest roots was found to have slight error so the upper 5% and 1% points were tabulated for small integral values of one parameter and values of the second parameter ranging up to 1,000.

P. S. Dwyer (Ann Arbor, Mich.)

947:

Tryon, Robert C. Domain sampling formulation of cluster and factor analysis. Psychometrika **24** (1959), 113–135.

The basic data of the multidimensional analysis under consideration are the intercorrelations between variables X_i and X_j, with $i, j = 1, \cdots, n$. In the words of the author the objective is "to determine and measure the smallest number, k, of dimensions that reproduce the correlations . . .". The progressive stages in the analysis, in which decisions by the analyst are required, are described. Possible alternative decisions are indicated in Table 1 which shows that many cluster and factor analysis methods are special cases stemming from decisions made at different stages of the general method. Critical examination of the resulting methods is presented and emphasis is given to the concept of dimension as applied to each.

P. S. Dwyer (Ann Arbor, Mich.)

948:

Gaffey, William R. A consistent estimator of a component of a convolution. Ann. Math. Statist. **30** (1959), 198–205.

Suppose one has a sample of size n from a population whose distribution function F is the convolution of G and Φ, where Φ is a normal distribution function with mean zero and known variance σ^2. The author proposes an estimate of the unknown distribution function G. The estimate is

$$\hat{G}_n(x) = \sum_{k=0}^{n} (-1)^k F_n^{(2k)}(x, h_n)(\sigma^2 t_n/2)^k/k!,$$

where $F_n^{(2k)}(x, h_n)$ is the $2k$th difference quotient with h_n the difference interval. The asymptotic mean square error of $\hat{G}_n$ as $n \to \infty$ with $t_n \uparrow 1$, $h_n \downarrow 0$ is obtained.

M. Rosenblatt (Providence, R.I.)

949:

Box, G. E. P.; and Lucas, H. L. Design of experiments in non-linear situations. Biometrika **46** (1959), no. 1/2, 77–90.

Let $y = f(X_1, X_2, \cdots, X_k; L_1, L_2, \cdots, L_p)$ be a known non-linear function of the k variates $X_1, X_2, \cdots, X_k$ containing p unknown parameters $L_1, L_2, \cdots, L_p$. An experiment is to be designed by selecting levels of the variates. The problem is to obtain efficient estimates of the p parameters from observations of the values of y in N trials. The authors present arguments to support a choice of $N = p$ (further replicating the entire set to achieve desired accuracy). It is assumed that preliminary estimates of the p parameters are available, that f is approximately linear in the neighborhood of these values, that the experimental errors are normally and independently distributed with a common variance. Graphical and numerical discussions are applied to illustrative examples. Various relaxations of the assumptions are briefly discussed. Reference is made to earlier related studies.

C. J. Maloney (Frederick, Md.)

950:

Archbold, J. W.; and Johnson, N. L. A construction for Room's squares and an application in experimental design. Ann. Math. Statist. **29** (1958), 219–225.

The authors are concerned with the following problem of T. G. Room [Math. Gazette **39** (1955), 307]: To arrange the $n(2n-1)$ two-element subsets formed from a set of $2n$ objects in a square array of side $2n-1$ such that each row (and each column) of the array has $n-1$ empty cells and n cells containing one each of n two-element subsets which partitition the set of $2n$ objects. The problem has a trivial solution for $n = 1$, no solution for $n = 2$ or 3, and a solution (given by Room) for $n = 4$. The authors find a cyclic solution for $n = 4^m$ (not $n = 2^{2m-1}$, as stated in the introduction) and they discuss two methods for using these Room designs in statistical analysis.

Room's problem is equivalent to a problem concerning a pair of loops, and the reviewer would like to sketch the connection before presenting the authors' construction. Let G^* consist of the first $2n-1$ natural numbers, $n > 1$, and let G consist of G^* with 0 adjoined. There is loss of generality in assuming that the Room design has the subset 0, x in the row-column position (x, x), for each x in G^*. This assumed, define two binary operations r, c on G as follows: (i) if x, y are distinct elements of G^*, the subset x, y appears in the row-column position (xry, xcy) of the design; (ii) $0r0 = xrx = 0c0 = xcx = 0$ and $0rx = xr0 = 0cx = xc0 = x$ for each x in G^*. Then (G, r), (G, c) are two commutative loops of order $2n$, exponent 2, with a common identity element 0, which are further restricted by the following condition: (C) if p, q are in G^*, the system of equations $xry = p$, $xcy = q$ has at most one (unordered) solution x, y if p, q are distinct and has no solution with x, y in G^* if p, q are equal. Conversely, if a "Room pair" of loops is understood to be a pair of loops satisfying the requirements of the preceding sentence, then every Room pair defines a Room design.

It is easy to see that if (G, r), (G, c) is a Room pair of order $2a$ and (H, r), (H, c) is a Room pair of order $2b$, the two pairs determine a "direct product" Room pair (K, r), (K, c) of order $4ab$ and an "indirect product" Room pair of order $(2a-1)(2b-1)+1$. For the latter type, we delete the identity elements of G and H, form a direct product and then re-introduce an identity element. These two processes lead from Room's original design of order 8 to Room designs of orders 50, 64, 344, 400, $\cdots$.

Now for the authors' construction. Aside from trivialities (they put 0, x on the anti-diagonal instead of the main diagonal of the square, and they use the language of geometry and of matrix theory) it is as follows: Let G be the Galois field of 2^k elements, $k \geq 2$. Define r, c by $xry = x + y$, $xcy = (x^{-1} + y^{-1})^{-1}$ for distinct elements x, y of G^*, and complete the definition as in (ii). Then (G, r), (G, c) is a Room pair precisely when the equation $x^2 + x = 1$ has no solution in G; that is, precisely when k is odd. Since $2n = 2^k$ and k is odd, then $n = 4^m$.

R. H. Bruck (Madison, Wis.)

951:

Bose, R. C.; and Mesner, Dale M. On linear associative algebras corresponding to association schemes of partially balanced designs. Ann. Math. Statist. **30** (1959), 21–38.

The following basic lemma on incidence matrices is proven: Let B_i ($i=0, 1, 2, \cdots, m$) be a set of symmetric incidence matrices of order v so that (a) $B_0=I_v$ (the unit matrix), (b) $\sum_{i=0}^{m} B_i=J_v$ (the matrix having each of its elements as unity), (c) $B_jB_k=\sum_{i=0}^{m} p_{jk}^{i}B_i$ ($j, k, =0, 1, \cdots, m$) for some set of constants p_{jk}^{i}. Then the B_i's are the association matrices of a PBIB design with m associate classes. The authors suggest some applications of this lemma in verifying the existence of a PBIB design with a given association scheme. Some results are derived concerning the characteristic roots and their multiplicities of the matrices $B=\sum_{k=0}^{m} c_kB_k$ and $P=\sum_{k=0}^{m} c_kP_k$, where c_k's are a set of real or complex coefficients and $P_k=(p_{ik}^{j})$ ($k=0, 1, \cdots, m$). These generalize the earlier results on PBIB designs with two associate classes by Connor and Clatworthy [same Ann. **25** (1954), 100–102; MR **15**, 494].

R. G. Laha (Washington, D.C.)

952:

Girshick, M. A. An extension of the optimum property of the sequential probability ratio test. Ann. Math. Statist. **29** (1958), 288–290.

Let $f(x, \theta)$ be a family of densities or discrete probability functions depending on a real parameter θ. Let T^* be a sequential probability ratio test for testing the hypothesis $H_0(\theta=\theta_0)$ against $H_1(\theta=\theta_1)$, $\theta_0<\theta_1$. The following extended optimum property of T^* is proved. Let $\bar{\theta}$ be a number lying between θ_0 and θ_1 such that, for $\theta\neq\bar{\theta}$, there is a unique non-zero root, say $h(\theta)$, of

$$E_\theta\left[\frac{f(x, \theta_1)}{f(x, \theta_0)}\right]^{h}=1$$

and that $h(\theta)>0$ for $\theta<\bar{\theta}$ and <0 for $\theta>\bar{\theta}$. Further assume that for each θ there is a θ' such that

$$\left[\frac{f(x, \theta_1)}{f(x, \theta_0)}\right]^{h(\theta)} f(x, \theta)=f(x, \theta').$$

If T is any sequential procedure such that $L(\theta, T)\geqq L(\theta, T^*)$ for $\theta<\bar{\theta}$, and $L(\theta, T)\leqq L(\theta, T^*)$ for $\theta>\bar{\theta}$, then $E_\theta(n|T)\geqq E_\theta(n|T^*)$ for all $\theta\neq\bar{\theta}$.

G. Kallianpur (East Lansing, Mich.)

953:

Tingey, F. H.; and Merrill, J. A. Minimum risk specification limits. J. Amer. Statist. Assoc. **54** (1959), 260–274.

Let x denote the true value for a numerical characteristic of an item of product, and let $y=x+\epsilon$ denote an observed value on x for the same item. Let further λ_t and μ_t denote "technical limits" for x and λ_s and μ_s denote "specification limits" for y. An item is classified as nondefective if the corresponding y falls into (λ_s, μ_s), and is classified as defective otherwise. Misclassification occurs when (*) for an item with x in (λ_t, μ_t) y falls outside (λ_s, μ_s) or (*) for an item with x outside (λ_t, μ_t) y falls into (λ_s, μ_s).

Assuming x and ϵ to be independently and normally distributed with means μ and 0 the authors determine, for given μ, λ_t, μ_t and variances of x and ϵ, and for a variety of assumptions about the losses associated with (*) and (**) those values of λ_s and μ_s for which the expected total loss due to misclassification is minimized. Formulas and tables for facilitating computation are given.

D. M. Sandelius (Göteborg)

954:

Owen D. B.; and Wiesen, J. M. A method of computing bivariate normal probabilities. Bell System Tech. J. **38** (1959), 553–572.

Charts and formulas are given by means of which probabilities of the type Prob $(X>h, Y>k)$ can be rapidly computed for correlated random variables X and Y with a joint normal distribution. The method is applied to some problems of choosing best specification limits with respect to technical limits when testing is subject to random error. [Cf. the preceding review].

D. M. Sandelius (Göteborg)

955:

Pompilj, Giuseppe. Su la regressione interamente pseudo-lineare. Fac. Sci. Statist. Demogr. Attuar. Ist. Statist. Ist. Calcolo Probab. Publ. no. 30 (1957), 18 pp.

Let X be a random point in the affine vector plane. If the regression of every linear functional fX with respect to every other gX is linear, then X is said (by the author) to be of entirely linear regression (OELR). If, for each fX, gX, and n, the polynomial $p(fX)$ of degree n in fX that minimizes $E[p(fx)-gX]^2$ is linear, then X is said to be of entirely pseudo-linear regression (OEPLR). Assuming that all moments exist, and that the covariance structure of X is non-singular, the author gives characterizations of OEPLR. For example, the distributions of all fX are similar if and only if X is OEPLR. Assuming absolute continuity, the author shows that the density of X has circular symmetry about the mean with respect to the covariance metric when the fX are similarly distributed; this implies (under absolute continuity) that OEPLR is equivalent to OELR, which is known not to be true if the "entirely" is dropped. The author overlooks the usual argument by means of Fourier transforms that makes the assumption of absolute continuity superfluous.

L. J. Savage (Rome)

956:

Madansky, Albert. Least squares estimation in finite Markov processes. Psychometrika **24** (1959), 137–144.

Consider a Markov chain with a states and stationary transition probability matrix. Let

$$M=(m_{ik}; i=1, \cdots, a; k=1, \cdots, n-1),$$

$$N=(m_{ik}; i=1, \cdots, a; k=2, \cdots, n),$$

where m_{ik} is the proportion observed on trial k in state i. Let $T=(t_{ij}; i, j=1, \cdots, a)$, where t_{ij} is the transition probability of going from state j to state i. A usual estimate of T is given by $\bar{T}=NM'(MM')^{-1}$. Let N_i, T_i be the ith rows of N, T. The author proposes the estimate $\hat{T}_i=N_i\hat{\Lambda}_i^{-1}M'(M\hat{\Lambda}_i^{-1}M')^{-1}$ of T_i where $\hat{\Lambda}_i$ is the diagonal matrix with diagonal elements $m_{i,k+1}-\sum_{j=1}^{a}\bar{t}_{ij}^{2}m_{j,k}$. He shows that the covariance matrix of $\hat{T}_i-T_i$ is asymptotically smaller than that of $\bar{T}_i-T_i$.

M. Rosenblatt (Providence, R.I.)

NUMERICAL METHODS

See also 705, 828, 936, 1127, 1219, 1226, 1237.

957:

Ehrlich, Louis W. Monte Carlo solutions of boundary value problems involving the difference analogue of $\partial^2u/\partial x^2+\partial^2u/\partial y^2+Ky^{-1}\partial u/\partial y=0$. J. Assoc. Comput. Mach. **6** (1959), 204–218.

In order to find the solution of the differential equation in the title which assumes prescribed values on the boundary of the square $0<x<1$, $0<A<y<A+1$ the author replaces the differential equation by the simplest approximating five point difference equation in a square grid of mesh length 1/20 and solves the resulting finite difference problem numerically by means of random walks. The computations were performed on the ORDVAC at the Aberdeen Proving Ground. The accuracy of the results turns out to be in agreement with theoretical results obtained by J. H. Curtiss [*Proceedings, seminar on scientific computation*, 1949, pp. 87–109, IBM Corp., New York, 1950; MR **13**, 286] and W. Wasow [J. Res. Nat. Bureau Standards **46** (1951), 462–471; MR **13**, 960].

W. Wasow (Madison, Wis.)

958:

Vladimirov, V. S.; and Sobol', I. M. Computation of the least eigenvalue of the Peierls equation by the Monte Carlo method. Vyčisl. Mat. **3** (1958), 130–137. (Russian)

The equation is

$$n(P)=\lambda\int a(P')h(P')\times\frac{\exp\left\{1-|P-P'|\int_0^1 a[tP+(1-t)P']dt\right\}}{4\pi|P-P'|^2}n(P')dP'.$$

A stochastic process is described and two examples given.

A. S. Householder (Oak Ridge, Tenn.)

959:

★Kopal, Zdeněk. Operational methods in numerical analysis based on rational approximations. On numerical approximation. Proceedings of a Symposium, Madison, April 21–23, 1958, pp. 25–43. Edited by R. E. Langer. Publication no. 1 of the Mathematics Research Center, U.S. Army, the University of Wisconsin. The University of Wisconsin Press, Madison, 1959. x+462 pp. (1 insert) $4.50.

In this paper the author develops the approximation of non-polynomial difference and differential operators by means of ratios of polynomial operators. Such operator functions as $\log(1+\Delta)$, $\sinh^{-1}\delta/2$, are expanded in continued fractions form and the Padé approximants for small orders indicated.

Application of these operators is made to derive numerical integration formulas. The author states that these formulas, applied to the numerical solution of ordinary differential equations, have many advantages over other methods such as the Adams-Bashforth. It seems clear that many new algorithms will be generated through the attack proposed. *P. C. Hammer* (Madison, Wis.)

960:

★Salzer, Herbert E. Some new divided difference algorithms for two variables. On numerical approximation. Proceedings of a Symposium, Madison, April 21–23, 1958, pp. 61–98. Edited by R. E. Langer. Publication no. 1 of the Mathematics Research Center, U.S. Army, the University of Wisconsin. The University of Wisconsin Press, Madison, 1959. x+462 pp. (1 insert) $4.50.

In this paper certain criteria are stated as desirable for divided difference algorithms to be used in approximating functions of two variables by polynomials. Certain criteria suggested seem impossible of complete achievement. For example, for interpolation in two variables it is generally not possible to select points of interpolation arbitrarily but such sets of points must exhibit certain algebraic independence. The same remark applies to formulas involving partial derivatives (osculating formulas).

The author proposes two types of divided difference formulas with the concomitant interpolating series and discusses various properties of both types. Error analysis is deferred. Several problems for future investigation are proposed. *P. C. Hammer* (Madison, Wis.)

961:

★Schoenberg, I. J. On variation diminishing approximation methods. On numerical approximation. Proceedings of a Symposium, Madison, April 21–23, 1958, pp. 249–274. Edited by R. E. Langer. Publication no. 1 of the Mathematics Research Center, U.S. Army, the University of Wisconsin. The University of Wisconsin Press, Madison, 1959. x+462 pp. (1 insert) $4.50.

Since 1936 the author has dedicated a large part of his scientific activity to the study of his "variation-diminishing" operators. This basic notion turned out to be very efficient for discussing the oscillatory properties of functions of a single variable and, in particular, of the solutions of eigenvalue problems. The paper at hand investigates approximations of a given function $f(x)$ by ordinary or trigonometric polynomials from this point of view in the special case where the approximation is generated by a linear functional operator. At first the important result is established that the approximation of $f(x)$ by Bernstein polynomials $B_n(x)$ is variation diminishing. This leads to some very explicit information about the behaviour of this Bernstein approximation. If for instance $f(x)$ is convex (without being linear), then $B_n(x)$ is also convex and $B_n(x)>f(x)$. Moreover $B_{n-1}(x)>B_n(x)$ unless $f(x)$ is a polygonal line of special type. Analogous properties hold for the approximation of a periodic function $f(t)$ by the so called de La Vallée Poussin means. Both methods of approximation avoid therefore unnecessary oscillations of the approximating functions. As it is well known, such unnecessary oscillations may appear if approximation by least squares or in the sense of Tchebycheff is used. In the second part of his paper the author shows that "variation-diminishing" in the sense of his definition also leads under certain assumptions to a decrease of the total variation $T(f)$ of the given function. The result of Popoviciu $T(B_n)\leq T(f)$ is re-established, and it is shown that the equality sign holds in this relation if and only if $f(x)$ is monotone. This property is also shared by de La Vallée Poussin means. In his last chapter the author introduces variation-diminishing integral-operators of the convolution type and a wider class of "convexity-

preserving" kernels. A theorem due to Ch. Loewner permits the decision whether some special kernels are convexity preserving or not. *E. Stiefel* (Zürich)

962:

★Golomb, Michael. Approximation by functions of fewer variables. On numerical approximation. Proceedings of a Symposium, Madison, April 21–23, 1958, pp. 275–327. Edited by R. E. Langer. Publication no. 1 of the Mathematics Research Center, U.S. Army, the University of Wisconsin. The University of Wisconsin Press, Madison, 1959. x+462 pp. (1 insert) $4.50.

In this paper the problem of approximating a function f of N independent variables by functions of fewer variables is very seriously attacked. In part A least-square approximations are studied beginning with linear approximations of the following type: Let f be a given function depending on a "parameter" x_0 and N variables $x_1, x_2, \cdots, x_N$; to approximate f by an expression

$$\sum_{k=1}^{n} (a_{1k}u_k{}^1 + \cdots + a_{Nk}u_k{}^N),$$

where the a_{jk} are given functions of the parameter and $u_k{}^i$ is an unknown function of the single variable x_i ($i=1, 2, \cdots, N$; $k=1, 2, \cdots, n$). The problem is solved along the well-known lines of the least-square technique using some ingenious short-cuts for solving the Euler equations of the variational problem. The result may be described briefly in the following way. $u_k{}^i$ is obtained by integration with respect to the remaining variables $\neq x_i$ and linear transformations involving the inversion of given matrices. Generally the solution is not unique. More complicated is the least-square approximation in the case of "overlapping variables", that is, approximation by linear combinations of unknown functions depending on $(N-1)$ of the variables $x_1, x_2, \cdots, x_N$, the coefficients being given functions of a single variable x_i.

Better but also more complicated approximations are available, taking into account bilinear approximations. If f depends, for instance, on two variables x_1, x_2, a bilinear approximation of f has the form $u_1{}^1u_1{}^2 + \cdots + u_n{}^1u_n{}^2$, where $u_k{}^1$ are functions of x_1 and $u_k{}^2$ functions of x_2. The solution of the approximation problem is obtained by computing the so-called Schmidt functions of f. They are the solutions $u_j{}^1, u_j{}^2$ of the linear homogeneous system of integral equations

$$\int f u_j{}^1 dx_1 = K_j u_j{}^2, \quad \int f u_j{}^2 dx_2 = K_j u_j{}^1$$

to be solved for the largest number K_j for which such solutions exist. The author investigates also more general bilinear approximations involving given coefficient functions depending on a parameter and hybrids of linear and bilinear approximations, as for instance the following problem. Let f be a given function of x_1, x_2, x_3 and let $a^i(x_i)$ be given coefficient functions ($i=1, 2, 3$). A least-square approximation of f is wanted in the form

$$a^1u_1{}^2u_2{}^3 + a^2u_1{}^3u_2{}^1 + a^3u_1{}^1u_2{}^2 + a^1a^2u^3 + a^2a^3u^1 + a^3a^1u^2,$$

where $u_1{}^i, u_2{}^i, u_3{}^i$ are unknown functions of x_i. The solution of this problem is performed by computing certain Schmidt functions. This type of approximation may be well suited for numerical computation. These results on Schmidt functions are extended in the paper also to multilinear approximations. At the end of this first part of the paper the author proposes a method of successive approximations for computing Schmidt functions, this is to say, for solving the systems of integral equations mentioned above.

Part B of the paper is devoted to Tchebycheff approximations. The author restricts himself to linear approximations, and he begins with approximations of

$$f(x_1, x_2, \cdots, x_N)$$

by a sum of N functions $v^j(x_j)$ of a single variable. Let x be the general point of an N-dimensional cube having the coordinates $x_1, x_2, \cdots, x_N$. Then the deviation

$$\sup_{(x)} (f - v^1 - v^2 - \cdots - v^N)$$

must be minimized by appropriate choice of $v^1, v^2, \cdots, v^N$. Let $E(f)$ be the minimal deviation. As it is well known from the general theory of Tchebycheff approximation, lower bounds for $E(f)$ are available. (Any choice of $v^1, v^2, \cdots, v^N$ yields of course an upper bound.) The author introduces for this purpose "projection cycles". They are certain sets of points in the cube having the property that a weighted mean of the corresponding values of f furnishes a lower bound of $E(f)$. This method is also extended to other cases of linear approximation.

In the last chapter of the paper an algorithm is described for constructing a minimizing sequence of approximating sums $(v^1+v^2+\cdots+v^N)$ converging to the solution of the Tchebycheff problem. The case $N=2$ is worked out explicitly; in the general case some additional difficulties are discussed. Some ideas of part B are based upon an earlier paper of S. P. Diliberto and E. G. Straus [Pacific J. Math. **1** (1951), 195–210; MR **13**, 334].

As the author points out, functions of several variables are a major obstacle in numerical work. Let us hope therefore that his important theoretical investigations will soon become a source for efficient numerical procedures.

E. Stiefel (Zürich)

963:

Filin, A. P. Interpolation polynomials for functions of several variables. Prikl. Mat. Meh. **21** (1957), 678–688. (Russian)

The author develops explicit relationships, in terms of determinantal recurrence formulas, for transferring from the Lagrangian equal-interval interpolation form of a polynomial into the standard coefficient description. He sketches the treatment for a polynomial in one variable, and then describes the extension of this process to a polynomial in several variables.

In particular, he presents tables used directly in calculating the coefficients from the functional values at equal intervals, for polynomials of up to degree four. He then carries through a computation, using these tables, for a polynomial in three variables of degree two, determining the 27 coefficients in terms of the values at a cubic, equally-spaced grid of 27 lattice-points in three space. The major contribution here appears to be the technique and tables for explicitly performing this translation from values to coefficients. *J. W. Carr, III* (Chapel Hill, N.C.)

964:

Kogbetliantz, E. G. Computation of Sin N, Cos N and $\sqrt{N}$ using an electronic computer. IBM J. Res. Develop. **3** (1959), 147–152.

Rational approximations in the form of continued fractions are derived for the computation of the sine and cosine functions to 10 decimal digits in 4 operations (i.e. multiplications and divisions) or 18 digits in 7 operations. Using the same approach to obtain a first approximation for use with Newton's iterative method yields (on a binary machine) square roots to 12 decimal digits in 3 operations and cube roots to 8 digits in 4 operations or 15 digits in 5. *B. A. Chartres* (Sydney)

965:

Goldstine, H. H.; Murray, F. J.; and von Neumann, J. The Jacobi method for real symmetric matrices. J. Assoc. Comput. Mach. **6** (1959), 59–96.

In 1846 C. G. J. Jacobi invented a direct and constructive method for proving that a symmetric matrix can be diagonalized by orthogonal transformations. He uses as elementary transformations 2-dimensional rotations in planes spanned by 2 coordinate axes. Or—geometrically speaking—the ellipsoid represented by the given matrix is introduced and its plane section by such a coordinate plane is examined. Then the 2 corresponding coordinate axes are rotated into the principal axis of this plane section. Observing that the sum of the squared elements of the matrix is invariant under orthogonal transformations he shows that the sum τ of the squared off-diagonal elements is diminished by each rotation and vanishes in the limit. Later on, Runge recommended this method as a numerical procedure for computing eigenvalues of symmetric matrices. The paper under consideration is an extensive study of such a procedure taking into account convergence properties and rounding-off behaviour as well. In the introduction the authors discuss the numerical difficulties inherent in all methods for computing the characteristic polynomial of a given matrix and of finding the eigenvalues by computing the roots of this polynomial. Therefore direct methods—for instance Jacobi's method—should be preferred. They state the problem, introduce a sequence $B^{(i)}$ of matrices obtained by groups of Jacobi rotations and give an estimate of the deviations of the diagonal elements of $B^{(i)}$ from the wanted eigenvalues based upon an estimate of the sum τ_i corresponding to $B^{(i)}$. They show that the Jacobi algorithm can be established in such a way that $\tau_i \leq \exp(-i)$ and that the diagonal elements of $B^{(i)}$ are within $\exp(-i/2)$ of the true eigenvalues.

But the outstanding feature of the paper is a thorough discussion of the rounding-off properties of the algorithm. The authors use the same tools as in the famous paper on inverting of matrices of high order [Bull. Amer. Math. Soc. **53** (1947), 1021–1099; MR **9**, 471] by J. von Neumann and H. H. Goldstine. They deal with pseudo-operations of a computing instrument and discuss in particular the pseudo-square root and pseudo-trigonometric functions as components of the Jacobi algorithm. This enables them to give at the end of their paper explicit estimates of the rounding-off errors to be expected by computing eigenvalues using Jacobi's method. Very few algorithms in the field of numerical analysis have been studied so thoroughly as is Jacobi's method in this paper. *E. Stiefel* (Zürich)

966:

Hsu, L. C. An efficient process of successive approximation for solving algebraic or transcendental equations. Acta Math. Acad. Sci. Hungar. **9** (1958), 291–297.

The author considers the solution of the equation $f(x)=0$ for f real or complex and of class C''. He describes the well-known rule of false position as if it were novel. He gives convergence results and establishes the rate of convergence of the method.

H. H. Goldstine (Yorktown Heights, N.Y.)

967:

Hämmerlin, Günther. Zur numerischen Integration periodischer Funktionen. Z. Angew. Math. Mech. **39** (1959), 80–82.

Der Verfasser zeigt, daß bei der numerischen Integration einer periodischen Funktion über eine volle Periode mit Hilfe der Trapezregel die gleiche Fehlerabschätzung gilt, die man für Quadraturformeln höherer Ordnung (z.B. Simpson) mit denselben—äquidistanten—Stützstellen erhält, selbst wenn bei diesen noch die Ableitungen mitberücksichtigt werden. *F. Stallmann* (Giessen)

968:

Greenwood, R. E.; Carnahan, P. D. M.; and Nolley, J. W. Numerical integration formulas for use with weight functions x^2 and $x/\sqrt{(1-x^2)}$. Math. Tables Aids Comput. **13** (1959), 37–40.

In this note the authors calculate points and constants for quadrature formulas for the weight functions indicated on the interval $(-1, 1)$. The Chebychev method of requiring equal Christoffel numbers is used. As a result, for the weight functions $x(1-x^2)^{-\frac{1}{2}}$ several formulas are calculated with 6 and 8 points, respectively. With $2m$ as the number of evaluation points the table for the weight function $x(1-x^2)^{-\frac{1}{2}}$ is given to 8D with $m:1(1)4$. For weight functions x^2 and with n equal to the number of points of evaluation, tables are given for $n:1(1)6$. No error terms are given, although it is stated that for certain functions the error will be less than that of the Newton-Cotes type formulas with the same number of points. Quadrature formulas for the weight function x^n; $n:1(1)6$ on [0, 1] have been calculated by Fishman [Math. Tables Aids Comput. **11** (1957), 1–9; MR **19**, 177].

P. C. Hammer (Madison, Wis.)

969:

Dieulefait, Carlos E. New principles in the problem of mechanical quadrature. An. Soc. Ci. Argentina **166** (1958), 23–25. (Spanish)

The author considers quadrature formulae of Gauss type and presents an informal argument leading to a simple relation between the quadrature coefficients (Christoffel numbers), the corresponding weight-function, and the asymptotic distribution of the abscissae. He gives formulae for the cases of (i) Legendre, (ii) Hermite abscissae. Those for (ii) are incorrect: the error results from the use of an imperfect analogy in a previous paper [Giorn. Ist. Ital. Attuari **17** (1954), 36–46; MR **17**, 962].

H. P. Mulholland (Exeter)

970:

Hammer, Preston C.; and Stroud, Arthur H. Numerical evaluation of multiple integrals. II. Math. Tables Aids Comput. **12** (1958), 272–280.

[Siehe auch Math. Tables Aids Comput. **11** (1957), 59–67; MR **19**, 323.]

Die Verfasser geben eine Zusammenstellung praktisch brauchbarer Formeln zur numerischen Integration über

gewisse Gebiete eines Enklidischen Raumes der Dimension $n>1$. Diese Formeln gelten exakt für Polynome der n Variablen vom Grade $\leq k$. Eine Tabelle enthält Stützstellen und Gewichte für folgende Fälle: Würfel und Kugel der Dimension $n\leq 3$, $k\leq 7$; n beliebig, $k\leq 5$. Simplex mit $n\leq 3$, $k\leq 3$. Einheitskreis ($n=2$) für $k\leq 15$.

F. Stallmann (Giessen)

971:

Mikeladze, Š. E. Quadrature formulas for multiple integrals with the greatest possible degree of accuracy. Soobšč. Akad. Nauk Gruzin. SSR. **18** (1957), no. 1, 3–10. (Russian)

Although the title of this paper indicates a treatment of "multiple integrals", the author actually deals only with repeated integrals, and the major part of the paper deals only with quadrature in one variable. The contribution of the paper is an algebraic method of assigning abscissas in the complex plane so as to increase the degree of precision. No weight factors are involved in this Gaussian-type integration procedure. The derivation is completely algebraic and does not involve orthogonal polynomials. The function $f(z)$ to be integrated is assumed regular in a region which contains the segment along which integration is to take place. The final result appears to be a generalization of the many Gaussian-type formulas into the complex plane for a rather restricted situation.

J. W. Carr, III (Chapel Hill, N.C.)

972:

★Fox, L. The numerical solution of two-point boundary problems in ordinary differential equations. Oxford University Press, New York, 1957. xi+371. $9.60.

The aim of the present book has been to summarize, on an elementary level and with the aid of many illustrative examples, the methods for numerical construction of the solutions of boundary-value problems which are expressible in terms of ordinary differential equations of orders one to four. The book is divided into 10 chapters, the first three of which are preparatory for its main topic, containing a re-capitulation of the elements of interpolation, differentiation, and integration in terms of finite tabular differences (Chapter 2), and of the solution of algebraic equations with application to relaxation techniques (Chapter 3). Chapters 4, 5, and 6 contain an expose of step-by-step methods for the construction of approximate solutions of the boundary-value problems defined by ordinary differential equations of the first four orders. Chapter 7 discusses the determination of eigenvalues of homogeneous boundary-value problems; Chapter 8 deals with the initial-value techniques for boundary-value problems; Chapter 9 surveys the precision of different methods; and Chapter 10 contains a description of some miscellaneous methods and techniques (including a limited treatment of non-linear eigenvalue problems). Additional notes and (rather selected) bibliographies conclude each chapter.

The book is well organized and printed, and its leisurely discursive style facilitates its reading for the non-initiated. It contains few if any original mathematical results, being intended as it is primarily for instruction and use of the practical computer working with desk-type machines. For the more serious student of the subject, its point of principal interest should be a detailed illustration of the iterative techniques (or, what Fox calls "difference-correction" method) for boundary-value problems, which have been developed by Fox and his colleagues in the past ten years and which are usefully summarized in this book.

Z. Kopal (Manchester)

973:

Hamming, R. W. Stable predictor-corrector methods for ordinary differential equations. J. Assoc. Comput. Mach. **6** (1959), 37–47.

W. E. Milne's so-called predictor-corrector method for the numerical solution of ordinary differential equations [W. E. Milne, *Numerical solution of differential equations*, Wiley, New York, 1953; MR **16**, 864] is not always computationally stable. The author gives a—partly heuristic—discussion of this situation. In the case of linear differential equations of first order with constant coefficients the behavior of the round-off errors can be completely analyzed, and the instability is seen to be caused by exponentially growing "extraneous" solutions of the approximating difference equation. The author proposes a modification of Milne's scheme for which this exponential growth of round-off errors does not occur.

W. Wasow (Madison, Wis.)

974:

Stoller, L.; and Morrison, D. A method for the numerical integration of ordinary differential equations. Math. Tables Aids Comput. **12** (1958), 269–272.

The solution of the initial value problem $y'=f(x, y)$, $y(x_0)=y_0$ satisfies the integral equation

$$y(x_0+h) = y(x_0)+\int_{x_0}^{x_0+h} f(x, y(x))dx.$$

The new feature of the method suggested by the authors is to replace the integral by an approximate expression based on unevenly spaced subintervals, such as Gauss's or Radau's quadrature formula. The values of $y(x)$ at these points of subdivision can be calculated by the Runge-Kutta method. The paper contains a short error analysis and a few numerical examples. The results obtained by the proposed procedure are significantly better than those yielded by the Runge-Kutta method alone.

W. Wasow (Madison, Wis.)

975:

Laasonen, P. On the simultaneous determination of several eigensolutions of a self-adjoint system of differential equations. Math. Tables Aids Comput. **13** (1959), 13–20.

This paper is concerned with eigenvalue problems of the form

$$\frac{d}{dx}u(x) = F(x)+\lambda G(x)u(x), \quad Au(a)+Bu(b) = 0,$$

where $u(x)$ is an n-dimensional vector and $F(x)$, $G(x)$, A, B are $n\times n$ matrices. The problem is supposed to be definitely selfadjoint in the sense of Bliss [Trans. Amer. Math. Soc. **44** (1938), 413–428]. The present paper is a continuation of two previous articles [Ann. Acad. Sci. Fenn. A. I. no. 195 (1955); no. 230 (1956); MR **17**, 489; **20** #1028] by the author concerning an iterative method of solution for such eigenvalue problems. A rule for the appraisal of round-off errors is established and a slight modification of the procedure is suggested in order to reduce the accumulation of these errors. A simple example illustrates the theoretical argument. The reader is assumed to be familiar with the author's previous work on this subject.

W. Wasow (Madison, Wis.)

976:

Milne, W. E.; and Reynolds, R. R. Stability of a numerical solution of differential equations. J. Assoc. Comput. Mach. **6** (1959), 196–203.

The classical Milne method for the numerical integration of ordinary differential equations [see W. E. Milne, Amer. Math. Monthly **33** (1926), 455–460], which for theoretical purposes is equivalent to Simpson's one-third rule of numerical integration, is known to suffer from weak instability in the sense of G. Dahlquist [Math. Scand. **4** (1956), 33–53; MR **18**, 338; see p. 36]. That is, in the integration of $y'=Gy$ ($G=$ const.) there appears a cancerous term of the form $(-1)^n \exp[-(G/3)x]$ which for $G<0$ and x sufficiently large can invalidate the desired solution. It is now proposed to quench the unwanted term by calculating every now and then a new value by Simpson's three-eighths rule and averaging it with the value obtained from Simpson's one-third rule. The effectiveness of this procedure is established analytically for the special equation mentioned above and experimentally by solving $y'=-y$, $y(0)=1$ in fixed point arithmetic for $0\leq x\leq 30$, using various integration steps.

P. Henrici (Los Angeles, Calif.)

977:

Sokolov, Yu. D. Sur une méthode de la résolution approchée des équations intégrales non linéaires à limites variables. Ukrain. Mat. Ž. **10** (1958), 419–433. (Russian. French summary)

The author uses his "method of averaging of functional corrections" to solve the integral equation

$$(1) \qquad y(x) = \phi(x) + \int_a^x K(x,t) f[x,t,y(t)]dt,$$

where K, ϕ and f are continuous, and f is Lipschitzian in y. The method was used previously by the author for linear integral equations [see e.g. same Ž. **9** (1957), 82–100; MR **19**, 687]. The first approximation y_1, and its average α_1, are determined from the equations

$$y_1(x) = \phi(x) + \int_a^x K(x,t) f(x,t,\alpha_1)dt,$$

$$\alpha_1 = \frac{1}{h}\int_a^{a+h} y(x)dx,$$

and for $n>1$, y_n and α_n are determined successively by

$$y_n(x) = \phi(x) + \int_a^x K(x,t) f[x,t,y_{n-1}(t)+\alpha_n]dt,$$

$$\alpha_n = \frac{1}{h}\int_a^{a+h} [y_n(x)-y_{n-1}(x)]dx.$$

Under suitable restrictions involving the bounds of the functions, the length of the interval, and the Lipschitz constant, the convergence of the method is demonstrated and an estimation of the error at the nth approximation is made. Eight examples are given with numerical results.

D. H. Hyers (Los Angeles, Calif.)

978:

Perlman, H. S.; and Robson, B. A. Dirac continuum radial wave functions. Austral. J. Phys. **12** (1959), 30–41.

The Dirac continuum radial wave functions for an electron in a Coulomb field can be expressed analytically. However, because of their complexity, the solutions for the mercury nucleus ($Z=80$) have been tabulated for several values of the momentum p and orbital quantum number l.

A parameter κ is defined so that $\kappa=-(l+1)$ when $j=l+\frac{1}{2}$, $=l$ when $j=l-\frac{1}{2}$, where j is the total angular momentum. Whenever possible, functions are introduced which depend only on $|\kappa|$.

All functions are tabulated for $|\kappa|=1$, 2, 3, 4, and $p=0.2$, 0.4, 0.6 and 0.8. *C. Froese* (Vancouver, B.C.)

979:

Learner, A.; and Robson, B. A. Irregular Coulomb wave functions. Austral. J. Phys. **12** (1959), 94–97.

The irregular Coulomb wave functions are the logarithmic solutions, $G_L(\eta,\rho)$, of the differential equation

$$d^2y/d\rho^2+\{1-2\eta/\rho-L(L+1)/\rho^2\}y = 0,$$

which act as cosine functions as $\rho\to\infty$.

Two functions g_L and g_L' have been tabulated for a range of parameters $\eta\leq 10$, $L=5$, 6, 10, and 11, $\Delta\rho=1$, and $\Delta\eta=1$. From these G_L and the derivative G_L' can be obtained. *C. Froese* (Vancouver, B.C.)

COMPUTING MACHINES

See also 1246.

980:

★Thüring, Bruno. Einführung in die Methoden der Programmierung kaufmännischer und wissenschaftlicher Probleme für elektronische Rechenanlagen. Teil 2: Automatische Programmierung dargestellt an der Univac FacTronic. Robert Göller Verlag, Baden-Baden, 1958. 169 pp. DM 30.00.

This book is the second volume of two by Thüring; the other (1957) is reviewed as MR **20** #1432. This volume departs from the previous "algorithmic" approach to concentrate on usage of the Univac I digital computer. The first half of the book gives the instruction code and programming details for that machine. The second half gives a very thorough account of a most important development in digital computer programming, the "GP Compiler" (Generalized Programming) developed by A. Holt and W. Turanski for the Univac I. This latter program is a translator and assembly program (compiler) that allows any program written for the machine to be filed away automatically in a library for later call in by future programs. The mechanics of tying together variables with different names in different programs is described as being performed automatically during the translation and assembly of a "main program" or algorithm. Although other features, including automatic symbolic address assignment, are present, the automatic library making feature is by far the greatest contribution of this procedure. The author gives a thorough account of programs written in this language and of their translation, and lists the library for the system at the time of publication.

J. W. Carr, III (Chapel Hill, N.C.)

981:

Korolyuk, V. S. On a method of programming. Dopovidi Akad. Nauk Ukraïn. RSR. **1958**, 1292–1295. (Ukrainian. Russian and English summaries)

This paper describes a symbolic algorithmic notation for description of digital computer programs that is different in one essential from the ALGOL-like languages of Western Europe and the U.S.A. The author proposes the use of a "Contents of" operator and the use of symbols in statements (formulas) to describe addresses only. He also proposes recursive use of this operator (which has been used, but very infrequently, in such compiler translators as the Share Assembly Program, USE Assembler, and the GAT translator in the United States) both for data (values) and functions (algorithms). Thus the contents of an address may be either a datum or the address of another algorithm. With this recursive type of notation he gives two examples: The sum of n numbers (coded in three statements), and computation and storage of the terms of an arithmetic progression (three statements). Although he states that automatization of such a procedure in an actual digital computer would not be difficult, there is no indication that a translator has been produced for a specific machine. As a result of his investigation he proposes a special accumulator in the hardware of machines to perform this addressing operation as well as other operations on addressing. A more thorough account of such a built-in procedure has been given by Schecher [Bericht Internat. Math.-Kolloq. Dresden, 1955, pp. 69–81, VEB Deutscher Verlag der Wissenschaften, Berlin, 1957; MR **19**, 583]. *J. W. Carr, III* (Chapel Hill, N.C.)

982:

Linskiĭ, V. S. Calculation of elementary functions on automatic digital machines. Vyčisl. Mat. **2** (1957), 90–119. (Russian)

This paper, based on the work of approximation by numerous mathematicians in Western Europe and the U.S.A., and extended to fit Soviet computer needs, collects into one place a large number of algorithms specifically for binary computers with both fixed and floating point arithmetic. The floating point algorithms are probably the most elaborate published. Most of the routines are expressed in digit-by-digit computational form (often new), Tschebycheff approximations, and in the appropriate cases by second-order (i.e., Newton-like) iteration methods. The algorithms described are, for floating point: division, square root, sine, cosine, tangent, cotangent, exponential, natural logarithm, arcsine, arctangent, and decimal-to-binary conversion and binary-to-decimal conversion. No round-off error analysis is included.

The fixed point routines are less thoroughly described, since these approximations were apparently written for the Strela, a digital computer with only floating-point arithmetic. Complete decimal and octal coefficients for Tchebycheff approximations to most of the functions listed above are also available for standard ranges.

J. W. Carr, III (Chapel Hill, N.C.)

983:

Gillis, J. An application of electronic computing to X-ray crystallography. Acta Cryst. **11** (1958), 833–834.

A programme for examining observed structure factors for significant Harker-Kasper inequalities has been run successfully on the digital electronic computer at the Weismann Institute (WEIZAC). As an incidental aid in the preparation of the programme, a polynomial approximation has been calculated to an atomic scattering factor of carbon. *Author's summary*

984:

Rappoport, M. G. Calculation of finite differences on punched-card machines. Vyčisl. Mat. **2** (1957), 120–145. (3 inserts) (Russian)

The authors describe methods of determining finite differences for functions using ordinary punched card tabulator equipment (mainly the Soviet-manufactured tabulator T-5). Very thorough programming diagrams are given for determining forward differences up to the seventh order, which are, unfortunately, of little use to users of general purpose stored program digital computers. The latter part of the paper describes similar calculations using the relay calculator EV80-3, one example of which has been installed at a computation center in Leningrad. With the latter machine, programs for up to fourth order differences are shown. One form of calculating these, given in table 2 of the paper, might prove useful with a general purpose machine. *J. W. Carr, III* (Chapel Hill, N.C.)

MECHANICS OF PARTICLES AND SYSTEMS

See also 770, 1205.

985:

★Ландау, Л. Д.; и Лифшиц, Е. М. Механика. [Landau, L. D.; and Lifšic, E. M. Mechanics.] Theoretical Physics, Vol. I. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 206 pp. 5.25 rubles.

The present volume is number one in the course on theoretical physics by the authors, to be completed in 9 volumes. The full set has been announced for publication in English translation.

The material covered is largely that which is traditional in a course on Newtonian mechanics at the level of the first-year graduate class, so that it does not seem necessary to give a detailed discussion of the various sections. However, since the pedagogical style adopted differs substantially from that of the usual American or English text, the reader may find it useful to see the list of chapters, with brief comments on points which seem most significant to the reviewer.

I. Equations of motion. II. Conservation laws. III. Integration of the equations of motion. IV. Collisions of Particles. V. Small Vibrations. VI. Motion of Rigid Bodies. VII. Canonical Equations of Motion.

The discussion commences at the level of Hamilton's principle and Lagrange's equations, without prior motivation. Only the simplest case of independent coordinates is treated explicitly, without consideration of constrained systems or non-holonomic systems. In addition to the usual topics of conservation of energy, momentum, and angular momentum, Chapter II includes a discussion of the virial theorem. The title of Chapter III may be misleading to a mathematician, since it suggests a general theory of integration whereas, in fact, only some of the usual special problems of particle mechanics, up to the two-body problem and Keplerian motion, are treated. In Chapter VI vectorial methods are employed to supplement the Lagrangian procedure. Chapter VII includes Hamilton's canonical equations, the Hamilton-Jacobi equation, and a brief consideration of the theory of adiabatic invariants. Throughout the text a number of

special problems are given with solutions. In a few cases only the problems and answers are given.

{The student who has had a good course in intermediate mechanics, up to Lagrange's equations, will find this text to be a very useful and clear statement of the major parts of classical mechanics. It is a little surprising to the reviewer that, with their emphasis on Lagrangian and Hamiltonian methods, the authors did not include a chapter on relativistic particle mechanics. From a wider point of view, it is somewhat disturbing to the reviewer that in this, as in other volumes of their course, the authors select carefully those parts of mathematical physics which are amenable to neat and precise formulation. While this is undoubtedly a great simplification for the beginning student, it should not be overlooked that the more important problems of principle in the interaction of mathematical and physical thought are almost wholly lost, so that the reader fails to gain any very clear notions either of the scope or the limitations of the theory.}

E. L. Hill (Minneapolis, Minn.)

986:

★Artobolevsky, I. I. On the theory of synthesis of mechanisms for the reproduction of certain kinds of algebraic and transcendental curves. Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 293–300. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi+362 pp.

A short survey about well-known mechanisms and the curves generated by them (cissoid, cardioid, conchoid, four-bar sextic, cycloid; inversor, converters).

O. Bottema (Delft)

987:

Meyer zur Capellen, W. Die Beschleunigungsänderung. I. Mitteilung. Ing.-Arch. **27** (1959), 53–65.

Betrachtungen über die dritte Ableitung des Weges nach der Zeit (hier Ruck genannt) in der ebenen Kinematik. Formeln für die Tangential- und Normalkomponenten des Rucks für einen Punkt, wobei die Flächengeschwindigkeit im Hodographen herangezogen wird. Momentane Ruckverteilung für die Ebene. Der Ruckpol. Der Ort derjenigen Punkte für welche die Normalkomponente des Rucks verschwindet ist ein Kreis. Der Ruck des Momentanpoles. Bedeutung der Kreisungspunktkurve und der Angelpunktkurve für diese Betrachtungen. Anwendung auf die Relativbewegung.

O. Bottema (Delft)

988:

Drahos, István; Hornyik, László; and Hosszu, Miklós. Solution of a tool geometrical problem. Magyar Tud. Akad. Mat. Kutató Int. Közl. **3** (1958), no. 1/2, 83–96. (Hungarian. Russian and English summaries)

The profile curves cut upon a helicoidal surface by various types of cutters are studied.

G. Kron (Schenectady, N.Y.)

989:

Litvin-Sedoĭ, M. Z. Quelques relations géométriques concernant les systèmes mécaniques à plusieurs degrés de liberté. Bul. Inst. Politehn. Iaşi (N.S.) **4** (8) (1958), 85–98. (Russian. French and Romanian summaries)

The author examines certain applications of linear algebra and the theory of matrices to the geometry of a finite set of moving coordinate trihedrals. The illustrating examples are taken from kinematics, the theory of mechanisms and the geometry of a gyroscope in a Cardan suspension.

E. Leimanis (Vancouver, B.C.)

990:

Arhangel'skiĭ, Yu. A. Single-valued integrals in the problem of the rolling of a sphere on the plane. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 3, 33–37. (Russian)

The author recalls two cases of the rolling sphere for which S. A. Tshaplygin has shown the problem to be reducible to quadrature. (1) The center of mass does not coincide with the geometric center of the sphere, while the ellipsoid of inertia is an ellipsoid of revolution whose axis passes through the center of mass. (2) The center of mass coincides with the geometric center and the ellipsoid of inertia is a general triaxial ellipsoid. He then shows that all cases which possess single-valued integrals of the equations of motion are reducible to case 2. The proof is based on an approach credited to W. W. Golubev.

F. T. Geyling (Summit, N.J.)

991:

★Yamamoto, Toshio. Stability criteria for and response function of forced vibration in nonlinear systems. Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 251–258. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi+362 pp.

This paper (hereafter referred to as "Y") concerns criteria for the stability of the solutions to the nonlinear equation

$$\ddot{x}+p^2x+\epsilon f(x,\dot{x}) = F\sin(\omega t), \quad F>0.$$

The Y criteria were asserted to hold under weaker restrictions than one given in a paper (hereafter referred to as "KP") by K. Klotter and the reviewer [J. Appl. Mech. **20** (1953), 9–12; MR **14**, 647]. The Y criteria (equations (25) and (26)) appear to be identical with and derived under identical restrictions as equations [26 a, b, c] in KP. The author may have overlooked this because the KP paper emphasized a somewhat more restricted criterion ([2] in KP) which could be represented graphically in a simple manner, while the others could not.

E. Pinney (Berkeley, Calif.)

992:

Schubart, Hans. Über die Grenzamplitude einer Klasse selbsterregter Schwingungen. Ing.-Arch. **27** (1959), 66–72.

The author obtains upper and lower bounds for the amplitude of the limit cycle for the self-excited vibrations governed by the differential equation

$$\ddot{q}-(\operatorname{sgn}\dot{q})\tfrac{1}{2}\beta(1-\alpha^2q^2)\dot{q}^2+\kappa^2f(q)=0 \quad (\beta>0)$$

with $f(q)=\sum_{l=0}^{k}C_lq^{2l+1}$ ($C_l\geqq 0$). The paper is an extension of the work of Klotter and Kreyszig [Ing.-Arch. **25** (1957), 389–403; MR **19**, 899], who previously considered this equation.

G. B. Warburton (Edinburgh)

993:

Epstein, Saul T. Harmonic oscillator wave packets. Amer. J. Phys. **27** (1959), 291–293.

Time-dependent operators are used to find harmonic oscillator wave packets which oscillate without change in shape. The results are in agreement with those of Senitzky and others. Some useful averages are computed for these packets and some comments are made on the classical limit. *Author's summary*

994:
Fraeys de Veubeke, Baudouin. Méthodes variationnelles et performances optimales en aéronautique. Bull. Soc. Math. Belg. **8** (1956), 136–157.

995:
de Moraes, A. Effects of the earth's oblateness on the orbit of an artificial satellite. An. Acad. Brasil. Ci. **30** (1958), 465–510.

This is essentially an expository article containing a survey of the equations governing the motion of a mass-point about an oblate spheroid, and of their approximate solutions for certain types of initial conditions—with application to the observed motion of artificial satellites in the gravitational field of the Earth. In the first part of the paper, the first-order disturbing effects of the terrestrial oblateness are considered, and the perturbations of the apsidal and nodal lines, as well as those of the inclination of the orbital plane, are determined to quantities of that order. The second half of the paper contains a discussion of the second-order ellipticity effects for orbits situated in the equatorial plane of the attracting spheroid, and a closed solution for the equatorial orbit is given in terms of the elliptic functions if squares of the terrestrial ellipticity are ignored. Lastly, it is shown that the relativistic apsidal advance of eccentric orbits of the Earth satellites is of the same order of magnitude as the cube of the terrestrial oblateness. *Z. Kopal* (Manchester)

996:
Niţă, M. M. Étude sur le mouvement des satellites artificiels dans un milieu résistant. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **9** (1958), 337–358. (Romanian. Russian and French summaries)

La force résistante est proportionelle à la vitesse. L'auteur transforme les équations dynamiques d'une façon élégante et démontre que le mouvement du satellite peut être considéré comme celui d'un point matériel dans un champ créé par un centre attractif de masse variable. Après une approximation convenable les équations deviennent intégrables par quadratures. L'orbite et la vitesse du planète sont étudiés. *O. Bottema* (Delft)

997:
Persen, Leif N. Satellitenbewegung mit Reibung. Z. Angew. Math. Mech. **38** (1958), 437–442. (English, French and Russian summaries)

The author considers two particular cases of motion of a satellite in an atmosphere with friction: (i) the friction is proportional to the square of the velocity and inversely proportional to the distance r from the center of mass of the Earth, and (ii) the friction is proportional to the velocity and inversely proportional to the square of the distance r. In both cases exact solutions in the form of elliptic spirals for the path of the satellite are obtained. Singer's assertion that due to friction the path of the satellite tends to become a circle before the satellite starts to approach the Earth is discussed in detail by means of the solution obtained in the case (ii). Finally these calculations are applied to compute the lifetime of "Sputnik I", the instant of entry of the satellite into the adiabatic atmosphere being considered as the end of it lifetime. *E. Leimanis* (Vancouver, B.C.

STATISTICAL THERMODYNAMICS AND MECHANICS

See also 1091, 1092.

998:
Balazs, Nandor L. Statistical mechanics of strings. Phys. Rev. (2) **113** (1959), 1178–1180.

The author studies the Brownian motion of an elastic string in a thermostat without the usual assumption that the string is so thin that its bending energy may be neglected. (This assumption leads to the paradoxical result that the average potential energy of a string element is infinite.) He takes into account the finite thickness of the string, and the consequent potential energy of bending, as well as the kinetic energy of rotation of the string elements. The average potential energy of the string is then found to be finite. *S. Katz* (New York, N.Y.)

999:
★Bearman, Richard J.; Kirkwood, John G.; and Fixman, Marshall. Statistical-mechanical theory of transport processes. X. The heat of transport in binary liquid solutions. Advances in chemical physics, Vol. I, edited by I. Prigogine, pp. 1–13. Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London; 1958. xi+414 pp. $11.50.

The heat of transport in a mixture of species interacting with central forces is calculated by expressing the heat flux due to a concentration gradient directly in terms of one- and two-particle distribution functions. To obtain the non-equilibrium two-particle distribution $c_{\alpha\beta}^{(2)}$ in configuration space, the authors postulate a diffusion equation for the pair current density $j_{\alpha\beta}^{(2)}$ which is a straightforward extension of the usual expression for the singlet current density in terms of diffusion coefficients and driving forces. Requiring the divergence of $j_{\alpha\beta}^{(2)}$ in pair space to vanish then gives a differential equation for $c_{\alpha\beta}^{(2)}$, which may be linearized and solved by expansion in spherical harmonics. The final expression for the heat of transport contains contributions due to the deviation of $c_{\alpha\beta}^{(2)}$ from the equilibrium pair distribution, as well as quasi-thermodynamic terms similar to those obtained by previous treatments. Some simplification of the general result is achieved if the mixture is assumed to be a regular solution. *S. Prager* (Brussels)

1000:
Bat', G. A.; and Zaretskii, D. F. Effective boundary conditions in the theory of neutron diffusion (a review). J. Nuclear Energy **9** (1959), 252–266.

This review describes methods for determining effective boundary conditions (EBC) which ensure that in the asymptotic region a solution of the neutron diffusion equation coincides with the solution of the corresponding

transport equation. We discuss EBC for mono-energetic neutrons on plane and on cylindrical surfaces of 'black' and of 'grey' bodies. Results are given for EBC in the case of cylinders of arbitrary section.

We consider the simplest problem of EBC determination for neutrons moderated in a medium of heavy atoms and with a scattering cross-section not dependent on energy. The review includes results obtained by workers both in the U.S.S.R. and abroad. *Authors' summary*

1001:

Marchuk, G. I.; and Mikhailus, F. F. Resonance absorption of neutrons in an infinite homogeneous medium. J. Nuclear Energy **9** (1959), 267–280.

We consider the problem of the slowing down of neutrons in an infinite homogeneous medium with strong resonance absorption and uniformly distributed neutron sources.

The solution of the adjoint equation gives the probability that a neutron of energy E escapes resonance capture during the process of moderation down to a certain limiting energy. The solutions of the original and the adjoint problem enable one to use a perturbation functional to take into account the influence of Doppler broadening of the resonance level in the resonance integral.

The methods developed are applied to the calculation of the collision density and the resonance integrals for the first level of ^{238}U ($E_0 = 6.7$ eV) in pure uranium and in UO_2. *Authors' summary*

1002:

Kraichnan, Robert H. Classical fluctuation-relaxation theorem. Phys. Rev. (2) **113** (1959), 1181–1182.

Suppose a conservative physical system which is in thermal equilibrium is described by the n variables $x_n(t)$. (If the system has a Hamiltonian the x's are the coordinates and conjugate momenta.) The equations of motion are $\dot{x}_n = X_n$; their solution satisfies Liouville's equation. Now subject the x_n's to an infinitesimal perturbation $u_n(t)$; i.e., $\dot{x}_n = X_n + u_n(t)$, $t > 0$. Write the perturbation δx_n in x_n as

$$\delta x_n = \sum_k \int_0^t \zeta_{nk}(t, t')u_k(t')dt'.$$

The matrix $\zeta_{nk}(t, t')$ is the infinitesimal-impulse-response matrix. Find its mean value averaged over an equilibrium ensemble of which the system under consideration is a member. This averaged matrix $g_{nm}(\tau)$ is a function of $\tau = t - t' > 0$ only. What can we say about this matrix? It cannot be arbitrary but must be of such form that particular perturbations u_n which do not disturb thermal equilibrium should result in perturbations which give the correct equilibrium values for the ensemble averages of functions of x. The author investigates the particular perturbation which couples the system to its thermostat. From this he derives as the condition to maintain thermal equilibrium that $g_{mn}(\tau) = \Phi_{mn}(\tau)/kT$. $\Phi_{mn}(\tau)$ is the ensemble average of $x_m(t)[\partial E/\partial x_n]_{at\,t-\tau}$. E is the total energy of the system. If E is a sum of squares (the system is equivalent to a set of harmonic oscillators) this result is equivalent to the one obtained by Callen and Greene [same Rev. **86** (1952), 702–710; MR **14**, 230].

N. L. Balazs (Chicago, Ill.)

ELASTICITY, PLASTICITY

See also 770, 998, 1113, 1114, 1115.

1003:

★**Pearson, Carl E. Theoretical elasticity.** Harvard Monographs in Applied Science. No. 6. Harvard University Press, Cambridge, Mass., 1959. xi+218 pp. $6.00.

This book provides a discussion of some modern methods in the theory of elasticity. The author has chosen his topics carefully so that there is little overlap with existing books and his work is a useful addition to the literature. The style is clear and the emphasis is on general results rather than on detailed examples.

An account of vector theory, suffix notations and Cartesian tensors is contained in the first two chapters. Although vectors are introduced geometrically in Chapter 1, it is slightly confusing to find that some definitions seem to involve analytical ideas. In Chapter 2 a vector is regarded as a Cartesian tensor of the first order and it might have been clearer if all the vector theory had been based upon this, particularly the theory of gradient, divergence and curl. Subsequent chapters are written almost entirely in Cartesian tensor notation. A straightforward account of the analysis of stress and deformation is given in Chapters 3 and 4, and Chapter 5 deals with the basic equations of linear elasticity, mainly for an isotropic body.

The second half of the book contains interesting material which is not readily available elsewhere. Chapters 6 to 9 are concerned with linear elasticity, again mainly for isotropic bodies, and are probably the most important chapters in the book. They deal with general solutions, variational and function space methods, thermoelasticity and time dependent problems all of which are subjects of present day research. The final Chapter gives a brief account of nonlinear elasticity but conveys little idea of the developments which have taken place in this part of the subject during recent years. It does, however, contain a valuable exact formulation of the theory of elastic stability. *A. E. Green* (Newcastle-upon-Tyne)

1004:

★**Bhagavantam, S. Third order elasticity.** Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 25–30. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi+362 pp.

1005:

Williams, M. L. The stresses around a fault or crack in dissimilar media. Bull. Seismol. Soc. America **49** (1959), 199–204.

The equations of plane strain elasticity are solved in polar coordinates, for different materials in $\pi > \theta > 0$, $0 > \theta > -\pi$, with free boundary conditions on $\theta = \pm\pi$, continuity on $\theta = 0$ and given loading at infinity.

J. W. Craggs (Newcastle-upon-Tyne)

1006:

Lebedev, N. N.; and Uflian, Ia. S. Axisymmetric contact problem for an elastic layer. J. Appl. Math. Mech. **22** (1958), 442–450 (320–326 Prikl. Mat. Meh.).

The authors consider the distribution of stress in an

elastic layer resting on a rigid plane, when the upper surface of the layer is deformed by having pressed into it a rigid body of revolution. The equations of elastic equilibrium are reduced by the use of Hankel transforms to a pair of dual integral equations of the type

$$\int_0^\infty A(\lambda)J_0(\lambda r)d\lambda = f(r) \quad (r < a),$$

$$\int_0^\infty \frac{\lambda A(\lambda)}{1-g(\lambda)} J_0(\lambda r)d\lambda = 0 \quad (r > a).$$

An important feature of the paper is the development of a method of solving a pair of equations of this type by representing $A(\lambda)$ by an expression of the form

$$A(\lambda) = [1-g(\lambda)] \int_0^a \phi(t) \cos \lambda t dt$$

and showing that $\phi(t)$ is a solution of a Fredholm integral equation with a continuous symmetrical kernel, which is easily calculated in terms of the Fourier cosine transform of the function $g(\lambda)$. The procedure is illustrated fully by considering the special case in which the rigid indenter is a circular cylinder with plane base.

I. N. Sneddon (Glasgow)

1007:

Zeuli, Tino. Sforzi e deformazioni in uno strato del suolo, poggiante su un piano orizzontale rigido e sollecitato superiormente da un carico distribuito su un'area rettangolare. Riv. Mat. Univ. Parma **8** (1957), 59–72.

The equations of elastic equilibrium in the absence of body forces are solved by means of double Fourier transforms to provide the distribution of stress in an elastic layer resting on a rigid plane $z=c$. It is assumed that the free surface $z=0$ of the elastic layer is free from shearing stress and is subjected to a prescribed normal pressure. The analysis is purely formal, no attempt being made to evaluate (either algebraically or numerically) the integrals obtained by use of the Fourier inversion theorem.

I. N. Sneddon (Glasgow)

1008:

Johnson, K. L. The effect of spin upon the rolling motion of an elastic sphere on a plane. J. Appl. Mech. **25** (1958), 332–338.

"The motion and deformation of an elastic sphere rolling on an elastic plane are examined for the case when the sphere, in addition to its straight rolling motion, has an angular velocity of 'spin' Ω about an axis normal to the plane. The action of spin is to twist the area of contact. Surface tractions resulting from this rotation are found which demonstrate the necessity of partial slip in the area of contact. Previous investigations suggest that this slip cannot occur at the leading edge of the contact circle, so that a system of tractions is found which corresponds to zero stress at the leading point. It is shown that such a system of tractions gives rise to a transverse creep of the sphere in the direction of its rotation Ω. The magnitude of this creep is calculated for small values of Ω, when slip occurs to only a small extent. Experiments have been performed using a simple thrust bearing with plane parallel races. As the bearing rotates, the balls creep radially outward in the predicted manner. Quantitative measurements of this creep agree with the theoretical estimate over a wide range." (Author's summary)

J. Heyman (Cambridge, England)

1009:

Johnson, K. L. The effect of a tangential contact force upon the rolling motion of an elastic sphere on a plane. J. Appl. Mech. **25** (1958), 339–346.

"The motion and deformation of an elastic sphere rolling on an elastic plane under a normal contact pressure N have been studied for the case where a tangential force T is also sustained at the point of contact. Provided that $T < \mu N$ (μ = coefficient of friction), the sphere rolls without sliding but exhibits a small velocity relative to the plane, termed "creep". Following the work of Mindlin and Poritsky, it is shown that creep arises from slip over part of the area of contact, and further, that this slip takes place toward the trailing edge of the contact area. On the assumption of a locked region in which no slip occurs, of circular shape, tangential to the circle of contact at its leading point, surface tractions are found which satisfy the condition of no slip within the locked region and are approximately consistent with the laws of friction in the slip region. The variation of creep velocity with tangential force is thereby determined. Experimental measurements of the creep of a steel ball rolling on a flat steel surface are in reasonable agreement with the theoretical results." (Author's summary)

J. Heyman (Cambridge, England)

1010:

Vodička, Václav. Ein durch allgemeine Massenkräfte beanspruchtes unendliches elastisches Medium. Z. Angew. Math. Mech. **39** (1959), 2–8. (English, French and Russian summaries)

1011:

Collins, W. D. Some stress distributions in an elastic solid bounded internally by a spherical hollow. Proc. London Math. Soc. (3) **9** (1959), 242–252.

1012:

Mişicu, M. On the solving of the spatial problem of the theory of elasticity. Applications to the theory of plates. Rev. Méc. Appl. 2 (1957), no. 1, 171–190. (15 inserts)

1013:

Milne-Thomson, L. M. Flexure. Trans. Amer. Math. Soc. **90** (1959), 143–160.

In the conventional formulation of the flexure problem for a beam [e.g., I. S. Sokolnikoff, *Mathematical theory of elasticity*, 2nd. ed., McGraw-Hill, New York–Toronto–London, 1956; MR **17**, 800; p. 198], the two non-vanishing shear stresses are obtained in terms of derivatives of a harmonic function f: $\tau_{13} = \partial f/\partial y$ + other terms; $\tau_{23} = -\partial f/\partial x$ + other terms; where these other terms can be written down explicitly. It follows that the complex combination $\tau_{13} - i\tau_{23}$ involves

$$\frac{\partial f}{\partial y} + i\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial x} + i\frac{\partial f}{\partial x} = i\frac{d}{dz}(f+ig),$$

where g is complex conjugate to f; consequently the complex shear stress is expressible in terms of an analytic function. Milne-Thomson obtains this result from first principles, and uses it in conjunction with conformal mapping into a unit circle to obtain the solution for flexure of a circle, cardioid, and lemniscate. Similar but less compact mapping methods are given by Sokolnikoff [loc. cit., p. 219].

C. E. Pearson (Cambridge, Mass.)

1014:

★**Hetényi, M. The infinite beam on equidistant supports and related problems.** Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 369–374. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

The paper is an elegant application of classical methods of structural analysis, use being made of the insensitivity of the behaviour of a typical span to conditions in remote spans, leading to a method of systematic and rapidly convergent reduction of the structure. The results do not contain approximations, and the method is applied to the infinite beam, to an infinite sequence of open continuous frames, and to an infinite sequence of closed frames.

J. Heyman (Cambridge, England)

1015:

Solomon, Liviu. A propos du centre de torsion. Com. Acad. R. P. Romîne **8** (1958), 1113–1118. (Romanian. Russian and French summaries)

L'auteur considère le déplacement d'une barre pour un couple donné et étudit la modification de la solution par une transformation du système des coördonnées rectangulaires. Il en résulte que la rigidité géométrique et le degré de torsion ne dépendent pas du choix des axes.

O. Bottema (Delft)

1016:

Dem'ianov, Iu. A. Self-modelling problems in the dynamic bending of plates. Soviet Physics. Dokl. **118** (3) (1958), 197–198 (669–670 Dokl. Akad. Nauk SSSR).

The author derives the equation for bending of plates in terms of variables $xt^{-1/2}$, $yt^{-1/2}$.

J. W. Craggs (Newcastle-upon-Tyne)

1017:

Bassali, W. A. The analysis of singularly loaded and rigidly clamped thin elastic slabs with curvilinear boundaries. I. Proc. Cambridge Philos. Soc. **55** (1959), 121–136.

Complex variable semi-inverse method is used to obtain in finite terms solutions for some singularly loaded thin clamped curvilinear plates. The conformal mapping on the unit circle is

(i) $Z = c\zeta/(1+m\zeta^{p+1})$ $(c > 0, |m| \leq \frac{1}{p})$,

(ii) $Z = c\zeta(1+m\zeta^{p})$ $(c > 0, |m|(p+1) \leq 1)$.

Degenerate cases of circular and infinite plate agree with known results. Simple results pertaining to $p=1$ and 2 in (ii) are obtained for the inverse of an ellipse and a dumbbell shape.

No numerical results are given.

B. R. Seth (Kharagpur)

1018:

Morozov, N. F. Nonlinear problems of the theory of thin plates. Vestnik Leningrad. Univ. **13** (1958), no. 19, 100–124. (Russian. English summary)

The existence of solutions of the set of coupled nonlinear equations of v. Kármán for large deflections of isotropic plates is proved using apriori estimates of the solution and its derivatives. A method of continuation with regard to a parameter proposed by Schauder and Leray is applied. To obtain the apriori estimates, the estimates of biharmonic Green's function and its derivatives are constructed, which are more exact than those of Vorovitch. The results are generalized to the case of a plate with variable rigidity. (After the author's summary.)

J. Nowinski (Madison, Wis.)

1019:

Johnson, M. W.; and Reissner, E. On the foundations of the theory of thin elastic shells. J. Math. Phys. **37** (1959), 371–392.

This paper is concerned with establishing a thin elastic shell theory by expanding the solutions of the differential equations of three dimensional elasticity in terms of powers of a suitable dimensionless parameter. The intention is to establish shell theories of controlled degrees of accuracy. Attention is confined here to a circular cylindrical shell of constant thickness with small rotationally symmetrical deformations. The first terms in the expansions correspond to the elementary theory based on the assumption that h/a is negligibly small in comparison with unity. Second and higher order terms represent the separate effects of (i) finite values of h/a, (ii) transverse shear stress deformations, (iii) transverse normal stress deformation. By introducing the characteristic axial length $(ah)^{\frac{1}{2}}$ it is possible to deal with the usual boundary layer phenomena, the axial length h being associated with a different class of boundary layer phenomena. The theory incorporates the consequences of the one boundary layer while being insensitive to the consequences of the other boundary layer.

L. S. D. Morley (Farnborough)

1020:

Szelągowski, Franciszek. The state of stress in a circular disc loaded by concentrated forces inside the disc region. Rozprawy Inż. **6** (1958), 515–526. (Polish. Russian and English summaries)

This paper is devoted to the derivation of equations determining the stresses at any point of a circular disc subjected to tension by two concentrated forces equidistant from the centre of the disc. *Author's summary*

1021:

Van Fo Fy, G. A. Bending of an elliptical plate. Akad. Nauk Ukraïn. RSR. Prikl. Meh. **5** (1959), 29–37. (Ukrainian. Russian and English summaries)

The author obtains a solution of the problem of bending of an elliptical plate with freely supported or clamped edges under an arbitrary transverse load. The solution of the problem is obtained in elliptical coordinates, employing the Mathieu function. *Author's summary*

1022:

Rivlin, R. S. The deformation of a membrane formed by inextensible cords. Arch. Rational Mech. Anal. **2** (1958/59), 447–476.

Idealized membranes are examined, these being formed from the curvilinear network constituted by two families of thin flexible inextensible cords lying in a smooth curved surface. The angles of intersection of cords of different families are assumed to be independent of position on the surface, and intersecting cords do not move relative to each other at points of intersection. This gives a membrane with two directions of inextensibility at each point, and with this assumption it is shown that in any deformation in which the surface remains smooth, the principal axes

of strain bisect the angles between the cords. General equations of equilibrium are formulated for large deformations of such membranes and the results specialised to the cylindrically symmetrical case, a number of cylindrically symmetrical deformations being examined in detail. The author also considers the perturbation problem in which a small deformation is superposed upon a known large deformation, and again specialises the theory to the case where the large deformation problem possesses cylindrical symmetry. The small deformation of a cylindrical membrane is considered as a special case of this and Fourier series solutions for specified edge displacements are derived.

The paper has an obvious bearing on important practical problems involving the deformation of fabrics and of systems of reinforcing cords embedded in rubber.

J. E. Adkins (Nottingham)

1023:

★Chattarji, P. P. Torsion of a large thick plate having transverse isotropy. Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 129–136. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi+362 pp.

Torsional stresses and displacements of a transversely isotropic large thick plate for which torsional shearing forces are prescribed on its boundary surfaces are obtained by expressing the applied shearing force in terms of a Fourier-Bessel integral. Two special cases are treated: (i) linear distribution within a circular area at the opposite boundary surfaces; (ii) concentrated torques on the opposite boundary surfaces. *R. M. Morris* (Cardiff)

1024:

★Narayana, G. L. Torsion of isotropic and composite materials. Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 137–144. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi+362 pp.

In this paper the torsion function for cylinders having certain cross-sections composed of one or two different isotropic materials have been obtained by using a general solution of Laplace's differential equation in polar coordinates. The following cross-sections are considered: (1) Simple cross-sections: circle, ellipse, equilateral triangle; (2) composite cross-sections; concentric circles, similar ellipses. *R. M. Morris* (Cardiff)

1025:

Teodorescu, Petre P. Sur le problème plan de l'élasticité de certains corps anisotropes. I. Relations entre tensions et déformations spécifiques. Com. Acad. R. P. Romîne **7** (1957), 395–400. (Romanian. Russian and French summaries)

In this paper the author reviews previous generalizations of Hooke's law and presents thereafter his own set of formulae. The author's stress-strain relations contain 36 elastic constants, amongst these constants there exist 15 independent relations; 21 constants are purely arbitrary. These relations are supposed to be related to a similar set obtained by Rabinovič [cf. Ob uprugih postoyannyh i pročnosti anizotropnyh materialov, Trudy Central'nyi aerogidrodinamičeskii institut, no. 582, 1946]. This report is not available to the present reviewer.

K. Bhagwandin (Oslo)

1026:

Teodorescu, Petre P. Sur le problème plan de l'élasticité de certains corps anisotropes. II. Corps à isotropie transverse. Com. Acad. R. P. Romîne **7** (1957), 401–406. (Romanian. Russian and French summaries)

The author reviews the different aspects of Hooke's law for bodies with transverse isotropy. Thereafter, he indicates practical methods for solving these types of problems for two plane cases, viz., when the axis of monotropy [cf. Behterev, Ž. Russ. Fiz.-Him. Obšč. **7** (1925), nr. 3–4] is orthogonal with respect to a defined plane, and when this axis is contained in the plane. It should be noted, however, that author's solution is merely of a formal character.

K. Bhagwandin (Oslo)

1027:

Teodorescu, Petre P. Sur le problème plan de l'élasticité de certains corps anisotropes. III. Les corps orthotropes. Com. Acad. R. P. Romîne **7** (1957), 503–508. (Romanian. Russian and French summaries)

The author reviews the different aspects of orthotropic bodies and puts forth some practical methods for solving the plane-type problems. These solutions can be considered as generalizations of the methods of Airy [cf. Philos. Trans. Roy. Soc. London **153** (1863), 49–80] and Marguerre [cf. Z. Angew. Math. Mech. **13** (1933), 437–438].

The solution obtained by the author is not directly amenable to numerical computation.

K. Bhagwandin (Oslo)

1028:

Teodorescu, Petre P. Sur le problème plan de l'élasticité de certains corps anisotropes. IV. Les corps à plan de symétrie élastique. Com. Acad. R. P. Romîne **7** (1957), 509–513. (Romanian. Russian and French summaries)

The method of analysis presented in part IV is similar to that of part III (reviewed above). In this paper the author considers the state of plane tension and plane deformation [cf. papers of Airy and Marguerre, vide sup].

K. Bhagwandin (Oslo)

1029a:

Teodorescu, Petre P. Sur le problème plan de l'élasticité de certains corps anisotropes. V. Les corps à un axe de symétrie élastique du troisième ordre. Com. Acad. R. P. Romîne **7** (1957), 641–647. (Romanian. Russian and French summaries)

1029b:

Teodorescu, Petre P. Sur le problème plan de l'élasticité de certains corps anisotropes. VI. Les corps avec un axe de symétrie élastique du quatrième ordre. Com. Acad. R. P. Romîne **7** (1957), 753–758. (Romanian. Russian and French summaries)

In parts V and VI the author reviews the different aspects of Hooke's law, related to bodies with an axis of symmetry of the third and fourth order, respectively. He indicates some practical methods of solving these types of problems. These solutions are, however, of a formal character [vide sup]. *K. Bhagwandin* (Oslo)

1030:

Borş, C. I. La relation entre les tensions et les déformations de flouage pour les corps orthotropes. Acad. R. P. Romîne. Fil Iaşi. Stud. Cerc. Şti. Mat. **9** (1958), no. 1, 125–130. (Romanian. Russian and French summaries)

On suppose que le matériel est homogène et orthotrope. De même, on suppose que la relation entre les tensions et les déformations de flouage est linéaire et que les déformations de flouage sont soumises au principe de la superposition des effets. *Du résumé de l'auteur*

1031:

Kaliski, Sylwester. Non-steady forced vibration of a rectangular parallelepiped. Arch. Mech. Stos. **10** (1958), 727–745. (Polish and Russian summaries)

The author describes the vibration of a rectangular parallelepiped subject to arbitrary bounded driving forces. The walls are assumed to be either rigidly fixed or free. The solution is based on a Green's tensor, giving the response to a unit impulse; a system of integral equations involving this tensor then determines the solution for a general force field. The author proposes an approximate means of solving these equations in terms of an associated set of algebraic equations. The paper is not self-contained, as many references are made to a previous paper by the same author [same Arch. **10** (1958), 329–370; MR **20** #5589]. *W. E. Boyce* (Troy, N.Y.)

1032:

★Greenspon, Joshua E. Flexural vibrations of a thick walled circular cylinder. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 163–173. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

This paper uses the three dimensional equations of elasticity to study the flexural vibrations of a hollow circular cylinder of finite length whose ends remain circular during the motion. The inner and outer surfaces are assumed to be stress free, while no radial or circumferential displacements occur at the ends. The author derives a frequency equation in the form of a sixth-order determinantal equation whose elements involve Bessel functions. The solution of this equation has been carried out numerically for the first two modes for various values of the ratios of inner to outer radius and length to outer radius. *W. E. Boyce* (Troy, N.Y.)

1033:

Ivovič, V. A. On forced pseudoharmonic oscillations of rods resting on elastic supports. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 42–45. (Russian)

The non-linear elasticity problem mentioned in the title, where the forcing term is a force given by

$$p = p_0 \sin(\pi x/l) \cos \Omega t,$$

leads to the equation

$$(1) \qquad \ddot{q} + \omega^2 q + \gamma q^3 + \kappa q d^2(q^2)/dt^2 = (p_0/m) \cos \Omega t.$$

The author considers the autonomous equation (2) obtained by replacing the second member by $p_0 q/mA$. (2) may be explicitly integrated by means of elliptic functions and it turns out that for certain ranges of the parameter values the second members of (1) and (2) are approximately equal, so that the solution of (2) may be considered as an approximate solution of (1). *J. L. Massera* (Montevideo)

1034:

Kazanceva, G. E. On the vibrations of circular plates of variable thickness. Akad. Nauk Ukraïn. RSR. Prikl. Meh. **4** (1958), 197–204. (Ukrainian. Russian and English summaries)

Taking a plate of uniform thickness as a subsidiary problem and making use of the reciprocity theorem, a Fredholm integral equation of the second type is obtained for the static deflection of a circular plate of variable thickness under a concentrated transverse load. The integral equation is further used as a subsidiary relation, in conjunction with the reciprocity theorem and the Laplace-Carson integral transformation, to arrive at an integral equation for transverse deflections of a circular plate of variable thickness in vibration. Application is made to a sector of an annulus. *W. W. Soroka* (Berkeley, Calif.)

1035:

Bogusz, Władysław. The stability of motion of bars rotating with variable angular velocity. Rozprawy Inż. **6** (1958), 547–555. (Polish. Russian and English summaries)

The solution of the equation of motion is obtained by means of Bessel functions. From an analysis of this solution it follows that for a fluctuating velocity there exist, besides the critical velocities, other velocities, for which the motion is not stable and that the stability depends on the frequency, and not the amplitude, of the velocity fluctuations. *From the author's summary*

1036:

Darevskiĭ, V. M.; and Kukudžanov, S. N. The stability of an orthotropic cylindrical shell subjected to torsion with internal pressure. Dokl. Akad. Nauk SSSR **123** (1958), 49–52. (Russian)

The problem of the stability of a thin cylindrical orthotropic shell of "medium" length under terminal torsional couples M and an internal pressure q is considered, the edges of the shell being freely supported or clamped. The basic equations consist of the equations of equilibrium taking the deformations of the shell into account, as well as of the usual relations between the strain and the internal forces, in terms of the displacement. If the system of the internal forces, in the basic state of equilibrium, does not depend on the non-dimensional coordinates φ and ξ, some simplifications corresponding to those of Donnell, for an isotropic shell, can be made. Thus an eighth order partial differential equation for the radial displacement is obtained. Critical values of the dynamical factors have been found for: (1) M and q acting separately; (2) "simple" simultaneous variation of M and q; and (3) M varying and q = const. *J. Nowinski* (Madison, Wis.)

1037:

Nutt, J. G. The critical load of a triangular framework when lateral buckling occurs. Appl. Sci. Res. A **8** (1959), 169–176.

"The critical load of a structure is that load at which

elastic buckling occurs. In this paper, the critical load of lateral buckling of a single triangle is evaluated, the assumption being made that the joints are fixed in position in space but free to rotate. The use of matrix algebra greatly simplifies the problem." (Author's summary)

J. Heyman (Cambridge, England)

1038:

★**Baker, Wilfred E.; and Allen, Frank J. The response of elastic spherical shells to spherically symmetric internal blast loading.** Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 79–87. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

1039:

Gazis, Denos C. Three-dimensional investigation of the propagation of waves in hollow circular cylinders. I. Analytical foundation. J. Acoust. Soc. Amer. **31** (1959), 568–573.

The equations of the linear theory of elasticity are solved to obtain the most general frequency equation for waves in an infinite tube with traction-free surfaces. The limiting forms are given for the cut-off frequencies corresponding to modes of plane strain, axial shear and torsion. The special cases of equivoluminal modes and of axially symmetric longitudinal and torsional modes are extracted from the general equation and discussed separately.

R. D. Mindlin (New York, N.Y.)

1040:

Gazis, Denos C. Three-dimensional investigation of the propagation of waves in hollow circular cylinders. II. Numerical results. J. Acoust. Soc. Amer. **31** (1959), 573–578.

In this paper are presented, in graphical form, the results of extensive computations of the roots of the frequency equation derived previously by the author [see the preceding review]. For Poisson's ratio 0.3, the frequencies of the first ten modes are given over a range of ratios of wall thickness to (real) wave length from zero to unity for four ratios of wall thickness to mean radius (including the solid bar) and for circumferential orders one and two. In addition, the cut-off frequencies are given for the entire range of ratios of wall thickness to internal radius. For the case of a thin-walled shell a comparison is made between the phase velocities of the first five modes and the phase velocities of the five modes of the Mirsky-Herrmann approximate equations [J. Acoust. Soc. Amer. **29** (1957), 1116–1123; MR **19**, 1000].

R. D. Mindlin (New York, N.Y.)

1041:

Waterman, P. C. Orientation dependence of elastic waves in single crystals. Phys. Rev. (2) **113** (1959), 1240–1253.

In certain crystals there are directions for which the orthogonal displacements of the three plane, elastic waves are parallel and perpendicular to the wave-normal. The author considers the effect of small deviations of the wave-normal, from these directions, on the phase-velocities, displacement directions and directions of propagation of energy. Explicit formulas are given for the cubic, tetragonal, hexagonal and trigonal crystal systems; and the phenomenon of internal conical refraction in cubic and trigonal crystals is discussed in detail. The author suggests numerous practical applications of his results to the field of measurement of properties of crystals by means of ultrasonic techniques.

R. D. Mindlin (New York, N.Y.)

1042:

Chakraborty, S. K. Disturbances in different types of elastic media. Geofis. Pura Appl. **35** (1956), 25–32.

The author considers the disturbances produced in an infinite medium when dynamical pressures are applied to the surface of a spherical cavity within it. Three special cases are considered: (a) The medium is assumed to be elastic and isotropic but to offer a resistance to the motion directly proportional to **u**, the displacement vector. By assuming that **u** can be expressed as grad ϕ, solutions are derived for the cases in which the applied radial pressure is periodic in the time, and is constant at $t=0$. (b) The medium is again supposed to be elastic but transversely isotropic about the radius vector; here the applied radial pressure is assumed to be impulsive. (c) The medium is assumed to display a certain type of linear viscoelasticity; only general remarks are made, no assumption being made about the precise nature of the applied pressure.

I. N. Sneddon (Glasgow)

1043:

Petrašen', G. I.; and Molotkov, L. A. On some problems of the dynamical theory of elasticity for a medium bounded by thin layers. Vestnik Leningrad. Univ. **13** (1958), no. 22, 137–156. (Russian. English summary)

A somewhat elaborate analysis is given of the theory of free vibrations in an elastic medium bounded by two parallel planes of unlimited extent. The treatment is based upon the general, three-dimensional, dynamical equations of elasticity, the thickness of the medium being regarded as a small parameter occurring in these equations. This type of procedure is familiar elsewhere in other investigations concerning elasticity theory of thin plates. Special attention is given to "low frequency" vibrations. According to the authors, it has been found possible on the present basis to discuss the "range of validity" of the ordinary engineering theory of vibrations of thin elastic plates.

H. G. Hopkins (Fort Halstead)

1044:

Kukudžanov, V. N. Shock waves in a compressible visco-plastic medium. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **11** (1958), no. 6, 61–72. (Russian. Armenian summary)

This paper presents a theoretical study of the problem of the dynamic expansion of a cavity, either spherical or cylindrical in shape, in a visco-plastic medium. The magnitude of the pressure applied at the surface of the cavity is supposed such that a shock wave occurs. The most significant feature of the investigation is that, within the framework of an apparently conventional theory, due attention is given to the occurrence of a discontinuous change in density, corresponding to the existence of a shock front. Some attention is given to definite physical situations, but few numerical results are given.

H. G. Hopkins (Fort Halstead)

1045:

Sankaranarayanan, R.; and Hodge, P. G., Jr. On the use of linearized yield conditions for combined stresses in beams. J. Mech. Phys. Solids **7** (1958), 22–36.

The authors show that if non-linear yield criteria are replaced by piece-wise linear approximations, great simplifications result in working actual engineering problems of plasticity. The results are applied to two examples of combined bending and torsion, and the theory can be extended to the most general case of combined bending in two planes, torsion, and axial force.

J. Heyman (Cambridge, England)

1046:

Bland, David R. The generalized plane strain of an elasto-plastic material. Z. Angew. Math. Phys. **10** (1959), 113–133. (German summary)

The defining equations are obtained for an elastic-plastic material under conditions of generalized plane strain, $w=Cz$. The material is linearly elastic with linear temperature effects up to the yield limit; thereafter it is either perfectly plastic or linearly work-hardening. All material properties are assumed independent of temperature and strains and rotations are all small.

For a rigid-perfectly plastic material in plane strain it is immaterial whether Mises or Tresca's condition is used; in either case the same three stress equations for σ_{xx}, σ_{yy}, and σ_{xy} are obtained. For the problem defined, there is a substantial mathematical difference. According to Mises criterion, the stress and displacements are coupled except in the exceptional case $\nu=0$, $T=C/\alpha$, whereas according to the Tresca criterion they will be uncoupled provided that

$$|(0.5-\nu)(\sigma_{xx}+\sigma_{yy})-E(C-\alpha T)| < k,$$

where k is the shear yield stress. Further attention is restricted to the Tresca criterion with the above inequality satisfied.

The perfectly plastic material is found to be governed by a set of hyperbolic equations. The characteristics and stress relations along the characteristics are the same as for the rigid-plastic material; the velocity relations are more complex. The governing equations for the work-hardening material are found to be elliptic; they are reduced to a set of two quasi-linear equations for the stress function ψ and shear angle ϕ.

The theory for a work-hardening material is illustrated with three simple examples in which the stress function depends upon only one variable.

P. G. Hodge, Jr. (Chicago, Ill.)

1047:

★Joga Rao, C. V. The elasto-plastic problem of a thin rotating disk with a central circular hole and hyperbolic thickness variation. Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 65–70. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi+362 pp.

A thin rotating disk is bounded by the surfaces $z=\pm Cr^n$. Assuming that the material is elastic/perfectly plastic and satisfies Tresca's yield condition, the author has found the complete elastic/plastic stress distribution in closed form for all speeds up to that at which the disk becomes fully plastic. It is assumed that $\sigma_\theta > \sigma_r > 0$ throughout; the assumption is shown to be valid in a particular example.

P. G. Hodge, Jr. (Chicago, Ill.)

1048:

Mikeladze, M. Weight and strength analysis of orthotropic rigid-plastic shells. Arch. Mech. Stos. **11** (1959), 17–31. (Russian. Polish and English summaries)

Optimum strength design of discs and shells is studied for a certain type of plastic anisotropy, which is a particular case of Hill's definition of plastically anisotropic materials [Proc. Roy. Soc. London Ser. A **193** (1948), 281–297; MR **10**, 83]. Established by Drucker and Shield [Quart. Appl. Math. **15** (1957), 269–281; MR **19**, 790] criterions concerning the rate of energy dissipation in minimum weight design have been applied to the considered types of anisotropic, rigid plastic sandwich structures.

The yield surface given in the paper represents only a surface passing through the intersection of yield hypersurfaces for plastic layers of a sandwich structure. In the particular case of simultaneous yielding at all points of a sandwich structure, that type of yield equation is justified but certainly it is not so in general. The flow law given in the paper does not correspond to the exact yield hypersurface but is valid under further restrictions concerning the scalar multiplier λ for the layers of a sandwich shell.

The paper deals also with the problem of optimal plastic non-homogeneity for a thick walled cylinder and a sphere, but gives only a trivial solution, since it assumes that the plastic non-homogeneity is defined by the elastic stress distribution.

A. Sawczuk (Warsaw)

1049:

Schumann, Walter. On limit analysis of plates. Quart. Appl. Math. **16** (1958), 61–71.

Extensions are made of the general plastic theory of plates developed recently by H. G. Hopkins [Proc. Roy. Soc. London Ser. A **241** (1957), 153–179; MR **19**, 342]. The author obtains results special to the various possible plastic régimes, and also discusses various types of boundary conditions and of discontinuities. The sole, detailed application made is to the determination of the yield-point load for an arbitrary convex, either simply-supported or built-in, plate subject to a concentrated transverse force; the exact value of this force is proved to be $2\pi M_0$ (where M_0 has the usual meaning). This result has been previously conjectured.

Solutions of limit analysis plate problems still remain extremely restricted, and this in spite of the fact that the nature of the governing equations is now well established [see H. G. Hopkins, loc. cit.]. Although this situation appears to be a reflection of inherent difficulties, further work along the lines considered in the present paper would be valuable.

H. G. Hopkins (Fort Halstead)

1050:

Zadoyan, M. A. Temperature stresses in infinite concrete slabs with account taken of creep. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **11** (1958), no. 1, 27–46. (Russian. Armenian summary)

Transient thermal stress in infinite concrete slabs with account taken of creep has been investigated on the basis of Arutyunyan's theory of creep. The instantaneous modulus of deformation is assumed to be time dependent. The integral equations of creep are reduced to differential equations with initial conditions. Two illustrative examples concern (a) harmonic oscillation of temperature of the surrounding atmosphere, and (b) symmetric cooling

of the faces. Numerical data show that creep halves the maximum thermoelastic stress in case (a), and changes the sign of stress in case (b). *J. Nowinski* (Madison, Wis.)

1051:

Rózsa, Pál; et Tassi, Géza. Calcul des constructions des barres statiquement indéterminées dans l'état élastique-plastique par le calcul des matrices. Magyar Tud. Akad. Mat. Kutató Int. Közl. **3** (1958), no. 1/2, 43–62. (1 plate) (Russian. Hungarian and French summaries)

1052:

★Michaels, J. E. Thermally induced elastic wave propagation in slender bars. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 209–213. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

The author considers the propagation of thermally induced longitudinal waves in a slender bar. The basic equations are solved by using the theory of Fourier transforms, Fourier cosine transforms for the case of the semi-infinite rod, finite Fourier cosine transforms for the case of a rod of finite length. The analysis is purely formal, only one specific example being treated and that an unrealistic one (in that the assumed temperature field is not a solution of the equation for the conduction of heat).

I. N. Sneddon (Glasgow)

1053:

★Deresiewicz, H. Solution of the equations of thermoelasticity. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 287–291. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

Writing the full coupled equations of dynamical thermoelasticity in terms of the displacement vector **u** and the temperature, the author uses Helmholtz's decomposition of a vector into a lamellar and a solenoidal part to obtain a fourth order equation for the scalar potential (corresponding to the absence of rotation). The solution of this equation is reduced to that of Helmholtz's form of the wave equation. The method is illustrated by discussing dilatational motion in an infinite solid, longitudinal motion of an infinite circular cylinder, and thickness-stretch motion in a slab of infinite radius but finite thickness. In each case the motion is shown to be dispersive as well as dissipative, the phase velocity decreasing and the energy loss increasing with a rise in frequency. In the case of thickness-stretch in a thick plate, numerical results are given for the motion of an X-cut of quartz.

I. N. Sneddon (Glasgow)

1054:

Sokołowski, Marek. The axially symmetric thermoelastic problem of the infinite cylinder. Arch. Mech. Stos. **10** (1958), 811–824. (Polish and Russian summaries)

The author splits the problem of determining the distribution of stress in an infinite circular cylinder due to a steady temperature field into two simpler problems: (a) the determination of stress in an infinite cylinder of radius R when nuclei of thermoelastic strain are distributed along the circle $r=\rho$, $z=0$; (b) the determination of stress in the same cylinder when heat is generated uniformly along the same circle. A complete analysis of both problems is given in terms of a thermoelastic displacement potential and Love's biharmonic function. The relevant equations are solved employing a Fourier transform and a finite Hankel transform to reduce the z- and r-coordinates. The convergence and asymptotic behaviour of the Fourier-Bessel series obtained by inverting the finite Hankel transform are thoroughly investigated for six special cases of problem (a). Full analytical details are given but no numerical results are reported. *I. N. Sneddon* (Glasgow)

1055:

Piechocki, Władysław. The stresses in an infinite wedge due to a heat source. Arch. Mech. Stos. **11** (1959), 93–109. (Polish and Russian summaries)

The author considers the thermal stresses set up in an infinite wedge when a constant heat source is located at a general point in the wedge. This provides a Green's function for the more general problem of an arbitrary distribution of thermal sources. The basic equations, which are expressed in terms of a thermoelastic displacement potential, are solved by means of Fourier series. The case of the semi-infinite plate is treated as a special example of the general theory. *I. N. Sneddon* (Glasgow)

1056:

Herrmann, G. On a complementary energy principle in linear thermoelasticity. J. Aero. Sci. **25** (1958), 660.

The author uses the formulation of linear thermoelasticity proposed by Biot [J. Appl. Phys. **27** (1956), 240–253; MR **17**, 1035]. He shows that, if two of the four basic equations are assumed, the other two are the Euler equations of a certain variational integral with respect to variations of the stress and the temperature. The assumed two equations correspond to Cauchy's law of mechanical equilibrium and Fourier's law of heat conduction. The derived two equations correspond to the stress-strain law and the law relating entropy, temperature, and stress. The variational theorem of the author generalizes the classical theorem of Castigliano.

W. Noll (Pittsburgh, Pa.)

1057:

Baltrukonis, J. H. Comparison of approximate solutions of the thermoelastic problem of the thick-walled tube. J. Aero/Space Sci. **26** (1959), 329–334.

The stress distribution in a thick-walled, circular cylindrical tube subjected to a steady-state temperature gradient across the tube wall is calculated, taking into account the variation with temperature of the elastic and thermal properties of the material. The calculations are made by five different methods: (I) by electronic digital computer, (II) by assuming an incompressible material, (III) by assuming a temperature-independent material and using the average properties of the material, (IV) by using the weighted-average properties of the material, and (V) by using a higher order approximation. Good agreement is found when high temperature levels do not persist over large portions of the tube. When the latter is the case, only method (V) gives much better than a qualitative picture of the stress distribution. *Author's summary*

STRUCTURE OF MATTER

See also 689, 983, 1041, 1145.

1058:

Nozières, P.; and Pines, D. Electron interaction in solids. Characteristic energy loss spectrum. Phys. Rev. (2) **113** (1959), 1254–1267.

The theory developed here is concerned with the inferences which can be drawn about the energy levels of electrons in solids from measurements on the transmission of fast electrons through thin films. The mathematical basis is essentially first-order quantum mechanical perturbation theory. The analysis is limited to non-relativistic energies, with the Coulomb repulsion of the incident electron and the electrons in the solid as the dominant perturbation.

The mathematical formulation of the theory is first reduced to the study of a formal "dielectric constant" (dielectric function), which is the analog of the ordinary optical dielectric constant, although distinct from it. Considerations similar to those underlying the Kramers-Kronig dispersion relations of optical theory are used to impose limitations on the real and imaginary parts of this function. The imaginary part of the dielectric function is related directly to the energy loss of the electrons transmitted through the film. The dielectric function is studied for various models of the electronic energy states in the solid, such as the free electron gas and the valence band approximation, with and without correlations among the electrons. The theory is related also to the (low-energy) Compton effect of photons passing through thin films. The paper contains a variety of physical considerations concerning this theory and its connection with other approaches, as well as with possible experimental measurements. *E. L. Hill* (Minneapolis, Minn.)

1059:

Callaway, Joseph; and Glasser, M. L. Fourier coefficients of crystal potentials. Phys. Rev. (2) **112** (1958), 73–77.

A method is developed for the calculation of the Fourier coefficients of the electrostatic potential of a given distribution of valence electrons in a solid. The valence electron wave functions are expressed as combinations of orthogonalized plane waves. The treatment takes full account of the nonspherical character of the atomic polyhedron. Application is made to lithium.

Authors' summary

FLUID MECHANICS, ACOUSTICS

See also 1140, 1188, 1189, 1190, 1191, 1192, 1210, 1219.

1060:

Aržanyh, I. S. On the equations of vortical motion of a polytropic gas. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **21** (1957), 29–34. (Russian)

Consider plane, steady, vortical two-dimensional motion of a gas obeying the equation of state $p = K\rho^n$, where K and n are constants, with $n \neq 1$. Let

$$H = \tfrac{1}{2}(u^2+v^2)+\frac{Kn}{n-1}\rho^{n-1}-U,$$

where u and v are the velocity components, and U is the potential of the body forces. The author obtains, by suitable eliminations, differential equations which may be used to determine (1) the stream function ψ and the density ρ, and (2) the function H and the vorticity ω. The three dimensional motion governed by the equations $[\mathrm{rot}\, v, v] = -\mathrm{grad}\, H$, $\mathrm{div}\,(\rho v) = 0$ is also considered from a similar point of view. *J. B. Diaz* (College Park, Md.)

1061:

Tomilov, E. D. Spiral motions of an ideal incompressible fluid. Tomsk. Gos. Univ. Uč. Zap. no. 25 (1955), 97–112. (Russian)

The author presents a method for investigating spiral motions of an ideal incompressible fluid. The method is based on the introduction of a stream function and a function similar in nature to a velocity potential. A number of spiral motions are shown to exist for an arbitrary choice of the ratio of angular to linear velocity as a function of one of the coordinate parameters.

M. D. Friedman (Needham Heights, Mass.)

1062:

★v. Krzywoblocki, M. Z.; Horiuchi, F. Y.; and Voda, J. J. On the similarity rules in diabatic flow. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 771–775. American Society of Mechanical Engineers, New York, 1958. xxvii + 864 pp. $20.00.

The flow considered is steady, small-perturbation flow of an inviscid perfect gas; in the vicinity of the body there is (small) heat addition. It is shown how such a flow can be related to a steady, incompressible, rotational flow; this constitutes a generalization of the Prandtl-Glauert relationships. Next, discarding the small-perturbation approximation, the treatment of two-dimensional flow by means of the hodograph transformation is considered. The existence of a related rotational, incompressible flow is again demonstrated. *W. R. Sears* (Ithaca, N.Y.)

1063:

★Smith, A. M. O.; and Pierce, Jesse. Exact solution of the Neumann problem. Calculation of non-circulatory plane and axially symmetric flows about or within arbitrary boundaries. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 807–815. American Society of Mechanical Engineers, New York, 1958. xxvii + 864 pp. $20.00.

The method, basically, is to solve numerically the integral equation for the source distribution at the boundaries by subdivision of the boundary and approximation to the source distribution function as step-wise constant. The problem is thereby reduced to solution of a set of simultaneous linear algebraic equations. The solution is "exact" in the sense that only numerical approximations are made. The authors suggest that as many as 300 simultaneous equations may have to be used. The method is tested by application to plane flow past a thick elliptic cylinder, a 45° wedge, a square step, and in a 90° elbow. Examples of axisymmetric applications include flow past an irregular blunt body (for which experimental data are available), over a body of revolution with (and without) area suction, and into an inlet in a stream. Two internal-flow constriction problems are also solved, including a sphere in a certain duct. *W. R. Sears* (Ithaca, N.Y.)

1064:

Kärrholm, Gunnar. A flow problem solved by strip method. Chalmers Tekn. Högsk. Handl. no. 208 (1959), 22 pp.

The problem considered here, the slow motion of a fluid past an infinite lattice of identical fixed circular cylinders, was suggested by the theory of flow through a porous medium.

It is reduced to finding a solution of the biharmonic equation in a certain finite region and solved numerically by analogy with a problem in the theory of plane stress in elasticity. The results are compared with a solution obtained by Shearer, using an unspecified electrical method, for a lattice of not insignficantly different dimensions. The agreement is stated to be reasonable.

K. Stewartson (Durham)

1065:

Elder, J. W. Steady flow through non-uniform gauzes of arbitrary shape. J. Fluid Mech. **5** (1959), 355–368.

The steady, two-dimensional, incompressible flow of a nearly uniform stream through a gauze which (a) is almost flat, (b) has nearly uniform properties, and (c) is almost normal to a parallel channel, is examined by assuming the flow inviscid except at the gauze; jump conditions are derived at the gauze by local application of relations significant for uniform plane gauzes inclined to a uniform stream. The smallness of the various non-uniformities is used to linearize the equations. A linear relation is obtained among up- and down-stream profiles, gauze shape and gauze parameters. Productions of a uniform shear and flow through linear and parabolic gauzes are solved explicitly. Data are given which validate the solutions well beyond the region of validity of the assumptions. The method is shown to apply also to axi-symmetric and diffuser flow and flow through multiple gauzes.

J. L. Lumley (Baltimore, Md.)

1066:

Popescu, I. L. Sur le mouvement non permanent d'un fluide à travers une grille de profils. II. Forces hydrodynamiques. Com. Acad. R. P. Romîne **8** (1958), 1003–1009. (Romanian. Russian and French summaries)

This paper gives the calculation of force components and moments to complete the work presented in Part I [Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **9** (1957), 121–137; MR **20** #1489].

W. R. Sears (Ithaca, N.Y.)

1067:

Söhngen, H.; und Meister, E. Beitrag zur Aerodynamik eines schwingenden Gitters. I. Z. Angew. Math. Mech. **38** (1958), 442–465. (English, French and Russian summaries)

An unstaggered infinite cascade of blades is placed in a stream of inviscid, incompressible fluid; the undisturbed velocity is parallel to the blade chords. The blades perform small harmonic oscillations, all of the same frequency but with amplitude and phase varying among the N blades representing a blade wheel. The first step in the solution is to relate the problem to one of steady flow through an unstaggered cascade of cambered blades. Essentially, it is possible to carry out this remarkable identification because of the periodic nature of the field of bound and free vortices in this problem. For the solution of this related, steady-flow problem a conformal mapping is introduced which transforms the cascade into an infinite line of blades ("horizontal cascade"); i.e., reduces the problem to a Hilbert boundary-value problem, for which known function-theoretical methods are available [e.g., N. I. Muskhelishvili, *Singulyarnye Integral'nye Uravneniya*, OGIZ, Moscow–Leningrad, 1946; Noordhoff, Groningen, 1953; MR **8**, 586; **15**, 434]. At this point it still remains to determine N constants that represent the strengths of the periodic vortex wakes. This finally involves a set of N linear algebraic equations; the solution is given.

No particular applications of the theory are made in the present paper. A review is provided of previous work on unsteady cascade flow, involving 12 references.

W. R. Sears (Ithaca, N.Y.)

1068:

Krasil'shchikova, E. A. Non-stationary motions of a wing of finite span in a compressible medium. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 3, 25–32. (Russian)

The author continues her investigations of non-stationary motion of a slender, slightly-cambered wing according to the method developed earlier. Solution of the problem is given in quadratures when fundamental wing motion is supersonic and neither tips nor trailing vortex system affect the wing surface. A finite span wing at a small angle of attack is treated herein. The paper is a more detailed version of an earlier paper [Dokl. Akad. Nauk SSSR **117** (1957), 777–780] and extends results from a paper on infinite span wing [Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1954**, no. 2, 25–41; MR **16**, 86].

M. D. Friedman (Needham Heights, Mass.)

1069:

Krasil'shchikova, E. A. Wing of finite span with a symmetrical profile in compressible flow. Soviet Physics. Dokl. **120** (**3**) (1958), 475–478 (51–54 Dokl. Akad. Nauk SSSR).

The case considered is that of a thin wing of arbitrary thickness distribution moving with variable speed and carrying out arbitrarily distributed oscillations in thickness. The solution by means of distribution of acoustic sources is formulated. The limits of integration are discussed for various cases.

W. R. Sears (Ithaca, N.Y.)

1070:

Keller, Joseph B. Surface waves on water of non-uniform depth. J. Fluid Mech. **4** (1958), 607–614.

A formal asymptotic solution of the exact linear theory for periodic gravity waves in water of arbitrary non-uniform depth is given. The solution is asymptotic in the sense that depth and wavelength must be small compared to the horizontal length scale of the bottom contour. (Depth and wavelength remain of the same order during the limit process.) It is found that in the first approximation the wave velocity at any point P is that which waves of the same period would have in water of uniform depth equal to that at P. Rays of wave propagation can be defined, and the flux of energy takes place along tubes of rays. These conclusions give a formal non-rigorous justification of methods which are familiar to workers in this field. However, the present method also yields higher approximations. Comparison is made with an exact solution for the uniformly sloping beach. The method fails when the depth is zero (where an infinite amplitude is predicted) and near caustics of the ray system (where the required modification is known).

F. Ursell (Cambridge, England)

1071:

Kranzer, Herbert C.; and Keller, Joseph B. Water waves produced by explosions. J. Appl. Phys. **30** (1959), 398–407.

Basing their analysis on the linear theory of surface waves, the authors find explicit formulas for the height of the water waves produced by an arbitrary axially symmetric initial disturbance in water of finite depth. The initial disturbance may be given as a distribution of impulse, or as an elevation or depression of the surface, or as a combination of these. The results furnish upper bounds for the wave amplitude when only the approximate size of the initial disturbance is known, but not its detailed shape. These results are applied to determine the wave resulting from the explosion of a typical large bomb above the surface. *D. H. Hyers* (Los Angeles, Calif.)

1072:

Yamada, Hikoji. Permanent gravity waves on water of uniform depth. Rep. Res. Inst. Appl. Mech. Kyushu Univ. **6** (1958), 127–139.

The author's numerical method, based on Levi-Civita's formulation of the non-linear wave problem was previously applied to waves in deep water [same Rep. **5** (1957), 37–52, 143–155; MR **19**, 487, 1118] and the solitary wave [ibid., **5** (1957), 53–67; **6** (1958), 35–47; MR **19**, 487; **20** #543] and is here applied to waves in water of finite uniform depth.

F. Ursell (Cambridge, England)

1073:

Reid, Robert O. Effect of Coriolis force on edge waves. I. Investigation of the normal modes. J. Marine Res. **16** (1958), 109–144.

A sloping beach can act as a wave guide for certain surface gravity waves (edge waves) travelling along the shore. The amplitude of these waves decays exponentially with distance from the shore. In the present paper the effect of the earth's rotation is examined. It is assumed that the pressure at any point is hydrostatic; bottom friction is neglected. Explicit expressions are found for the various modes. To each mode of a given wavelength along shore there now correspond 3 frequencies, instead of 1 when the earth's rotation is neglected. Two of these correspond to edge waves on a non-rotating earth; the phase speed of waves travelling to the right (facing shore from the sea) exceeds the speed of waves travelling to the left, and the difference is significant when the wavelength exceeds several hundred kilometers. The third mode is effectively a quasigeostrophic boundary wave (Kelvin wave) of very low frequency moving only to the left. The results of this thorough investigation are clearly set out in a summary but are too lengthy to be reproduced here. Specific examples are worked out in a paper by K. Kajiura [see the following review].

F. Ursell (Cambridge, England)

1074:

Kajiura, Kinjiro. Effect of Coriolis force on edge waves. II. Specific examples of free and forced waves. J. Marine Res. **16** (1958), 145–157.

The modification of edge waves due to Coriolis force [see the preceding review of R. O. Reid's paper] is examined in terms of two specific problems: dispersion of an initial deformation of the water surface; the forced wave due to an atmospheric pressure disturbance moving parallel to the coastline. Among other results it is found that the larger the scale of the disturbance the larger the effect of Coriolis force. In the northern hemisphere, the initial deformation of the water surface is split into two free edge waves in which the one moving to the left (facing the coast from the sea) has a larger amplitude than the one moving to the right. For forced waves it is found that a disturbance moving to the left is more effective than one moving to the right.

F. Ursell (Cambridge, England)

1075:

Wagner, Richard. Ein Strömungsproblem mit nicht eindeutig bestimmter Lösung. Z. Angew. Math. Mech. **38** (1958), 427–431. (English, French and Russian summaries)

This paper is concerned with the two-dimensional version of the following problem. A jet impinges normally on a straight wall with an aperture which faces the jet symmetrically. In general, part of the jet will pass through the aperture, while some of it will flow off along the wall. It is assumed that the ambient pressure, and hence the velocity at infinity are the same for the emerging and for the impinging jet. The problem is solved by means of Kirchhoff's method and the solution is expressed in terms of elliptic functions. For the case where the jet is slightly narrower than the aperture initially, the author finds two solutions of the kind described above, in addition to the undisturbed passage of the jet through the aperture. {It would be of interest to know whether these solutions are not in some cases unstable.} *A. Robinson* (Jerusalem)

1076:

Mimura, Yōichi. Phenomenological free streamline theory. I. Flow past a normal plate. J. Phys. Soc. Japan **13** (1958), 1361–1374.

Following L. C. Woods [Proc. Roy. Soc. London Ser. A **227** (1955), 367–386; MR **16**, 1061], the author constructs various potential flows with "free" streamlines along which the (empirically) given pressure distribution is given, through a hodograph.

G. Birkhoff (Cambridge, Mass.)

1077:

Riley, N. Effects of compressibility on a laminar wall jet. J. Fluid Mech. **4** (1958), 615–628.

In the boundary layer approximation, and assuming the viscosity proportional to (absolute) temperature, the author obtains similarity solutions for a compressible jet of air spreading out over a plane wall. The method of analysis is that of M. B. Glauert [same J. **1** (1956), 625–643; MR **19**, 87]. *G. Birkhoff* (Cambridge, Mass.)

1078:

Hasimoto, H. On the periodic fundamental solutions of the Stokes' equations and their application to viscous flow past a cubic array of spheres. J. Fluid Mech. **5** (1959), 317–328.

In this paper, periodic fundamental solutions for a viscous fluid moving slowly past a periodic array of spheres are given by use of Fourier series. As examples, the flow past a cubic lattice is examined for three types of lattice (simple, face-centered, body-centered) and the force acting on any one of the spheres is determined as a function of

volume concentration. It is found that the ratios of the values of this force to that acting on an isolated sphere are larger than unity and do not differ much for these three types if the concentration is small.

Y. H. Kuo (Peking)

1079:

Kynch, G. J. The slow motion of two or more spheres through a viscous fluid. J. Fluid Mech. **5** (1959), 193–208.

This paper deals with the problem of the motion produced by two or three spheres moving slowly under external forces through a viscous fluid. On the basis of Stokes' approximation, the solution for two spheres, when the distance between spheres is large, can be obtained by super-position of the solutions for a single sphere. The numerical results compare favorably with experimental data. A discussion of the interaction of three particles is also included. *Y. H. Kuo* (Peking)

1080:

Clarion, Claire; et Ghosh, Ambarish. Étude des actions dynamiques dans le mouvement oscillatoire forcé d'un ellipsoïde de révolution au sein d'un fluide visqueux. C. R. Acad. Sci. Paris **247** (1958), 634–637.

The boundary-layer theory is given for the small forced rectilinear oscillations of an ellipsoid of revolution in a viscous liquid. *W. R. Dean* (London)

1081:

Ghosh, Ambarish; et Clarion, Claire. Étude des actions dynamiques dans le mouvement oscillatoire forcé d'un ellipsoïde de révolution au sein d'un fluide visqueux. C. R. Acad. Sci. Paris **247** (1958), 676–679.

The results are given of experiments on small forced rectilinear oscillations in water of three ellipsoids of revolution and of a sphere. *W. R. Dean* (London)

1082:

Regirer, S. A. The influence of thermal effects on the viscous resistance of a steady uniform flow of liquid. J. Appl. Math. Mech. **22** (1958), 580–586 (414–418 Prikl. Mat. Meh.).

The author considers a layer of viscous fluid contained between two infinite flat plates, $y=0$ and $y=h$. The second plate moves in the x-direction with uniform velocity, U. The basic equations are three ordinary differential equations: (1) the defining relation between the stress, the viscosity, and the velocity; (2) the equilibrium relation; (3) the relation between the temperature, the stress, and the viscosity. This system is studied in detail for the case when the viscosity is an arbitrary function of temperature. An integral of this function is shown to be of significance in the determination of temperature and the asymptotic properties of a related integral are determined.

N. Coburn (Ann Arbor, Mich.)

1083:

Kuwabara, Shinji. The forces experienced by a lattice of equal flat plates in a uniform flow at small Reynolds numbers. J. Phys. Soc. Japan **13** (1958), 1516–1523.

Let h denote the lattice distance, **U** the velocity vector of approach parallel to the plates of length 2, and ω the vorticity. In Stokes' approximation, the title problem may then be formulated for the strip $0\leq y\leq h/2$: $\Delta^2\omega=0$; $\omega=0$ for $y=h/2$ and any x, and for $y=0$ and $|x|>1$. The disturbance stream function $\psi=\Psi-Uy$ satisfies $\nabla^2\psi=-\omega$; $\psi=0$ for $y=0$ and $y=h/2$, and $\psi_y=-U$ for $y=0$, $|x|<1$ (stick-condition). The Fourier transforms ω^* and ψ^* with respect to x are easily obtained in terms of g^*, which is the Fourier transform of the principal unknown function of the problem $g(x)\equiv\omega(0,x)$, $|x|<1$. Next, $\psi(x,y)$ is found in the form of an integral transform of $g(x)$ with a kernel in the form of an infinite series. Finally the stick condition yields for g the following integral equation:

$$\int_{-1}^{1} g(p)K(-2\pi|x-p|/h)dp = 2\pi U;$$

$$K(\gamma) = \log(1-\exp\gamma)-\gamma\exp\gamma/(1-\exp\gamma).$$

For large h an asymptotic solution is obtained by expansion of K and g in negative powers of h; for small h the equation for g may be approximated by an equation of the Weiner-Hopf type. Numerical results are given for the product of Reynolds number and drag coefficient, for overlapping ranges of h. *G. Kuerti* (Cleveland, Ohio)

1084:

Tamada, Kō; and Fujikawa, Hiroomi. The steady flow of viscous fluid at low Reynolds numbers passing obliquely through a plane grid made of equal parallel circular cylinders. J. Phys. Soc. Japan **14** (1959), 202–216.

An extension of the authors' calculations [Quart. J. Mech. Appl. Math. **10** (1957), 425–432; MR **20** #4988] to the case in which the velocity of approach is $U\mathbf{i}+W\mathbf{k}$, with **k** parallel to the cylinder axes and **i** forming the angle $\omega\neq\pi/2$ with the plane of the grid. Since no z-derivatives must occur, the equations have in Oseen's approximation the standard form with U as representative velocity, and the equation for w is not coupled with the pair for u and v. The form of $u-iv$ is therefore the same as in the paper quoted above; and, as for the non-analytic part of $u-iv$, the form of w is

$$W+\sum_s \exp kx_s \sum_m C_m K_m(kr_s)\exp im\vartheta_s.$$

In the present case, of course, the grid periodicity requires

$$z_s = r_s\exp i\vartheta_s = z-sh\exp i\omega,$$

where h is the distance between the cylinder axes.

Recurrence relations for the three sequences of coefficients, $A_1,\cdots,a_0,\cdots,C_0,\cdots$ are given; with approximate expressions for the first terms in the cases $ka\ll 1$, $kh=O(1)$ and $ka\ll 1$, $kh\ll 1$, where $k=U/2\nu$ and a is the cylinder radius. Numerical results for the force on the grid are supplied for $\omega=60°$. The flow is deflected toward the grid normal, and the velocity at downstream infinity is not $U\mathbf{i}+W\mathbf{k}$. *G. Kuerti* (Cleveland, Ohio)

1085:

Ladyženskaya, O. A. On the nonstationary Navier-Stokes equations. Vestnik Leningrad. Univ. **13** (1958), no. 19, 9–18. (Russian. English summary)

Miss Ladyzenskaya and others have proved a series of existence and uniqueness theorems for weak solutions of Cauchy's problem and the nonstationary Navier-Stokes equations. In E^3 the equation of uniqueness for any initial Reynolds number and all $t\geq 0$ remains open. Here the question is answered affirmatively for the case of two space variables, and some other propositions are proved.

Let Ω be a bounded region in E^3, and let $Q_T = \Omega \times [0, T]$, $T \geqq 0$. Theorem 1: The problem

$$(1) \qquad Lu \equiv \frac{\partial u}{\partial t} + \varepsilon \Delta^2 u - \nu \Delta u + u_k \frac{\partial u}{\partial x_k} + \operatorname{grad} p = f,$$

$$(2) \qquad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, \quad u|_{t=0} = u^0(x),$$

has a unique weak solution in Q_T if $\varepsilon > 0$, $\nu \geqq 0$, $f \in L_2(Q_T)$, $u^0 \in W_2^2(\Omega)$, $u^0|_{\partial\Omega} = 0$, and $\operatorname{div} u^0 = 0$; this solution satisfies the a priori estimates

$$(3) \qquad \int_\Omega u^2(x, t)dx + \int_0^t \int_\Omega \left[\varepsilon(\Delta u)^2 + \nu \sum_1^3 u_{x_i}{}^2\right] dxdt \leqq c_1,$$

$$(4) \qquad \int_\Omega \sum_1^3 u_{x_i}{}^2(x, t)dx + \int_0^t \int_\Omega \left[\varepsilon \sum_1^3 (\Delta u_{x_i})^2 + \nu(\Delta u)^2\right] dxdt \leqq c_2,$$

$$(5) \qquad \int_\Omega \left[\nu \sum_1^3 u_{x_i}{}^2(x, t) + \varepsilon(\Delta u(x, t))^2\right] dx + \int_0^t \int_\Omega u_t{}^2 dxdt \leqq c_3.$$

The constant c_1 depends only on $\int_\Omega u^2(x, 0)dx$ and $\int_0^t \int_\Omega f^2 dxdt$; c_2 depends only on ε, $\int_\Omega \sum_1^3 u_{x_i}{}^2(x, 0)dx$, and the preceding two integrals; and c_3 depends only on $\int_\Omega \{\Delta u(x, 0)\}^2 dx$ and all four of the previous quantities. Further, if u satisfies the inequality $\int_\Omega \sum_1^3 |u_i(x, t)^{3+\delta} dx \leqq c(\delta)$, $\delta > 0$, where $c(\delta)$ is independent of ε and t, then the set of solutions $u^\varepsilon(x, t)$ has a unique limit as $\varepsilon \to 0$ which is the weak solution of the boundary value problem for the Navier-Stokes equations.

The estimate (3) is obtained by integrating the scalar product of Lu with u over $\Omega \times [0, t]$. The second estimate (4) follows from (3) and the identity $\int_0^t \int_\Omega -Lu \cdot \Delta u dxdt = \int_0^t \int_\Omega -f \cdot \Delta u dxdt$ after its left member is integrated by parts. The third estimate is obtained from the last identity, but with Δu replaced by u_t. Existence can then be proved as in E. Hopf, Math. Nachr. **4** (1951), 213–231; MR **14**, 327]. Uniqueness is proved with the aid of (3) and Sobolev's inequality $\max_{x \in \Omega} u(x, t) \leqq c_4 F$, where $\int_\Omega [\Delta u(x, t)]^2 dx = F^2(t)$.

Next consider the system

$$(6) \qquad L_\rho u \equiv \partial u/\partial t - \nu \Delta u + u_{k_\rho} \frac{\partial u}{\partial x_k} + \operatorname{grad} p = f,$$

$$\operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0, \quad u|_{t=0} = u^0,$$

where u_{k_ρ} is the spherical mean with radius ρ of u_k. Theorem 2: Let $f \in L_2(Q_T)$, let $u^0 \in L_2(\Omega)$, and choose $\rho > 0$. The problem (6) has a unique weak solution in Q_T lying in $L_2(\Omega)$ for any t and having weak derivatives u_{x_k} in $L_2(Q_T)$.

The proof of the uniqueness of the solution of (6) in the case of the Navier-Stokes equations in two space variables is to appear in another paper.

N. D. Kazarinoff (Ann Arbor, Mich.)

1086:

Stewartson, Keith. On the free convection from a horizontal plate. Z. Angew. Math. Phys. **9a** (1958), 276–282. (German summary)

The equations describing the motion in the boundary layer over a horizontal plate are derived on the assumption that the boundary layers start at the edge of the plate and grow towards the centre. The author shows that for a heated plate facing upwards and for a cooled plate facing downwards, the equations do not admit of a solution, whilst for the cases of a heated plate facing downwards and a cooled plate facing upwards a numerical solution is obtained.

M. H. Rogers (Shrivenham)

1087:

Glomot, Christiane. Frottement et convection thermique dans le cas d'un écoulement laminaire le long d'une plaque plane non isotherme. J. Rech. Centre Nat. Rech. Sci. **1958**, no. 43, 81–92.

The laminar flow past a flat plate with a non-uniform temperature distribution is studied. In particular the plate is held at a constant temperature for a finite length followed by a short continuous transition region to a higher constant temperature. Profiles of the temperature and shear distribution are computed at various positions throughout the transition region. The results of using various hypothesis are discussed.

R. C. DiPrima (Troy, N.Y.)

1088:

Davies, D. R. Heat transfer by laminar flow from a rotating disk at large Prandtl numbers. Quart. J. Mech. Appl. Math. **12** (1959), 14–21.

The approximate method previously used for the flat plate by Davies and Bourne [same J. Mech. Appl. Math. **9** (1956), 457–467; MR **18**, 777] is extended to the problem of a heated rotating disk with an arbitrary radial distribution of surface temperature. A comparison of the numerical results for heat transfer with those calculated from an exact solution in the case of constant surface temperature by Millsaps and Pohlhausen [J. Aero Sci. **19** (1952), 120–126; MR **13**, 879] shows that very good accuracy is obtained when the Prandtl number is greater than about 6. (From author's summary)

D. W. Dunn (Ottawa, Ont.)

1089:

Corcos, Gilles M.; and Sellars, John R. On the stability of fully developed flow in a pipe. J. Fluid Mech. **5** (1959), 97–112.

All investigations of viscous flow in a pipe have tended to show that the flow is stable to infinitesimal axi-symmetric disturbances. That conclusion is unaltered by the present work which, however, points out the defective character of previous investigations. In particular, it is shown that two sets of modes exist, one of which has been overlooked before. This new set of modes is found to be the less stable of the two and to correspond to the set of unstable modes found for two-dimensional Poiseuille flow. For a given value of the wave-number and Reynolds number, the analysis yields only a finite number of eigenvalues and eigenfunctions. A difficulty arises, therefore, in attempting to treat an arbitrary initial disturbance; this difficulty is discussed briefly but not resolved.

W. H. Reid (Providence, R.I.)

1090:

Yih, Chia-Shun. Ring vortices generated electromagnetically. J. Fluid Mech. **5** (1959), 436–444.

The possibility of using electromagnetic forces to generate vortices analogous to Taylor vortices in a fluid between concentric cylinders is considered. When an electric current of density j_0 passes axially through a stationary fluid between cylinders of radii r_1 and r_2, it is shown that stability can be achieved by imposing along the axis of the cylinders a line current J such that $J \leqq -\pi j_0(r_2{}^2 - r_1{}^2)$ or $J \geqq \pi j_0 r_1{}^2$. For small differences in the radii r_1 and r_2, the problem is treated with effects of viscosity and finite conductivity included.

P. R. Garabedian (Stanford, Calif.)

1091:

Zwick, S. A. Behavior of small permanent gas bubbles in a liquid. II. Bubble clouds. J. Math. Phys. **37** (1959), 339–353.

[For part I, see same J. **37** (1958), 246–268; MR **20** #6870.]

A statistical model of a liquid-bubble mixture is given in terms of a distribution function representing the number of bubbles per unit of volume. This leads to a continuum theory of the motion of dense clouds of bubbles, and a system of partial differential equations governing motions of this type is written down. It is shown that the velocity of sound is decreased by the presence of a bubble cloud, more or less as if the liquid became more compressible. Applications of the theory are made to the study of a bubble cloud rising under the influence of gravity, and to oscillatory motions. *P. R. Garabedian* (Stanford, Calif.)

1092:

Zwick, S. A. Behavior of small permanent gas bubbles in a liquid. III. A forced vibration problem. J. Math. Phys. **37** (1959), 354–370.

The theory described in the preceding review is used to analyze experimental investigations of a liquid-bubble mixture in a vibrating tank. Problems concerned with the production of bubbles at the free surface of the liquid in the tank are discussed. Both a one-dimensional model and a three-dimensional model are used to describe resonance of the mixture and clustering of the bubbles.

P. R. Garabedian (Stanford, Calif.)

1093:

Münch, G.; and Wheelon, Albert D. Space-time correlations in stationary isotropic turbulence. Phys. Fluids **1** (1958), 462–468.

The time-correlation of the Fourier amplitudes, $E(k, \tau)$, is studied in the inertial range using the quasi-normality approximation to relate fourth-order mean values to second-order mean values. An initial value problem is considered for which it is assumed that $E(k, 0)$ is given by Heisenberg's equilibrium spectrum, and the resulting behavior of $[\partial^2 E(k, \tau)/\partial\tau^2]_{\tau=0}$ is then calculated numerically for four values of the Reynolds number. In the inertial subrange, this result leads to a correlation time $\tau_k \sim (v_0 k)^{-1}$, where v_0 is a velocity associated with the energy-containing range of wave-numbers. *W. H. Reid* (Providence, R.I.)

1094:

Squire, William. A unified theory of turbulent flow. I. Formulation of the theory. Appl. Sci. Res. A **8** (1959), 158–168.

It is suggested that the averaged Navier-Stokes equations do not form a determinate set for the solution of the mean flow problem because of the nature of the averaging process and that they may be made determinate by analysis and modification of this process. The new assumptions are (a) that $\overline{Df}/Dt = \overline{Df/Dt}$, where the averages are taken over volumes small compared with the scale of mean velocity variation but large compared with the eddy size, and (b) that the stress tensor is

$$\tau_{ij} = \delta_{ij}\bar{P} + (\varepsilon_{ij} + \mu)\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \rho\overline{u_i'' u_j''}$$

where ε_{ij} is an eddy viscosity arising from a random, discontinuous component of the motion and u_i'' is the remaining continuous part. The form and magnitude of ε_{ij} is to be inferred from the statistical theory of turbulence.

A. A. Townsend (Cambridge, England)

1095:

Ohji, Michio. On the theory of homogeneous axisymmetric turbulence. Rep. Res. Inst. Appl. Mech. Kyushu Univ. **6** (1958), 63–83.

The basic theory of homogeneous, axisymmetric turbulence is developed in some detail, applying the invariant theory of Cartesian tensors to discover the proper representations of the second- and third-order tensors occurring when the equations of motion are used to give the equivalent of the Karman-Howarth equation for isotropic turbulence. The forms of the double velocity correlation tensor and its Fourier transform, the energy spectrum tensor, are discussed in some detail, including their decomposition into components. The equivalent decomposition of an axisymmetric field of turbulence into quasi-isotropic, axial and transverse components is discussed and the possible difficulties of considering them as independent motions is emphasised. The material presented here is a very full account of the basic structure on which future attempts to develop the statistical theory must be based. *A. A. Townsend* (Cambridge, England)

1096:

Monin, A. S. The structure of atmospheric turbulence. Teor. Veroyatnost. i Primenen. **3** (1958), 285–317. (Russian. English summary)

This is an account of recent experimental and theoretical work on the structure of the turbulence in the earth's boundary layer. The layer is assumed to be in a condition of horizontal homogeneity and transferring constant shear stress and constant flux of total heat. It is therefore specified by these quantities and the characteristic length

$$L = -\frac{\tau^{3/2}}{\kappa(g/T_0)(Q/\rho c_p)}.$$

Experimental measurements of mean velocity distributions are in good agreement with the dimensional consequences of the specification for upward and downward heat fluxes. Also considered are the magnitudes of the turbulent velocity fluctuations and the balance between their production, transport and dissipation, and the temperature fluctuations and their balance. The correlations and spectra of the fluctuations within the Kolmogoroff range of similarity are described, and the theory of the space and time spectra in the inertial sub-range is treated in some detail. A short summary is given of work relating to pressure fluctuations, particle accelerations, turbulent diffusion and to the production of waves.

A. A. Townsend (Cambridge, England)

1097:

Otmahova, I. P. Drag of a hemisphere moving with constant velocity in a rarefied gas. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 4, 17–26. (Russian)

Problem of the drag of a hemisphere moving in a rarefied gas flow is solved when the hemisphere is hollow and the repeated impacts of the molecules being reflected from the concave surface is taken into account. The total drag is found as the sum of the drag on the convex and concave surfaces of the hemisphere.

M. D. Friedman (Needham Heights, Mass.)

1098:

Berezin, O. A. Some partial solutions of the equations of one dimensional unstable motion of a gas. Vestnik Leningrad. Univ. **14** (1959), no. 1, 145–149. (Russian. English summary)

The present paper states the exact solutions of the equations of plane, cylindrical spherical waves dependent on five arbitrary constants. These solutions are of the form

$$u = r\frac{\dot{f}}{f} + \frac{r^{-(k-1)(k+1)^{-1}N}}{1+(k-1)(k+1)^{-1}N} f^{1+(k-1)(k+1)^{-1}N}\frac{d\varphi}{dt},$$

$$P = r^{-2k(k+1)^{-1}N} f^{-k(1+(k-1)(k+1)^{-1}N)}\Phi(\xi),$$

$$\rho = r^{-2(k+1)^{-1}N} f^{-(1+(k-1)(k+1)^{-1}N)}F(\xi),$$

where $\xi = (r/f)^{1+(k-1)(k+1)^{-1}N} - \varphi$,

$$\int \frac{df}{(c_1+2c_2[(k-1)(N+1)]^{-1}f^{-(k-1)(N+1)})^{\frac{1}{2}}} = \pm(t+t_0),$$

φ, F and Φ are known functions. *Author's summary*

1099:

Glansdorff, P. La dynamique des fluides déduite de la moindre contrainte par la méthode variationnelle de Th. De Donder. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 236–239.

"Nous établissons par la méthode variationnelle de Th. De Donder, l'équation fondamentale de la dynamique des fluides à partir du principe de la moindre contrainte de Gauss, dans le cas général où les tensions dépendent des dérivés spatiales des vitesses jùsqu'à un ordre quelconque, et sans faire appel à la condition de conservation de la masse." (Du résumé de l'auteur) *Hirsh Cohen* (Delft)

1100:

Ting, Lu. Some integrated properties of solutions of the wave equation with non-planar boundaries. Quart. Appl. Math. **16** (1958), 373–384.

Generalising some earlier works [A. Ferri, J. H. Clarke, and L. Ting, J. Aero Sci. **24** (1957), 791–804; MR **19**, 799], the author establishes an integral formula for the lift on a non-planar surface whose generators are parallel to the direction of flow, in terms of the distribution of the normal velocity on the surface. The result is obtained by applying Green's theorem to a particular vector whose divergence is the left hand side of the linearized equation of supersonic flow (i.e. the wave equation of three variables). It is shown that the formula holds even if there are discontinuities in the normal velocity distribution such as may be expected in a wing-body combination. By obviating the need for the detailed calculation of the pressure distribution under certain conditions (no subsonic leading edges) the formula can be very useful in a number of cases.

A similar method is applied to acoustic diffraction problems in two dimensions. The case of diffraction as a rectangular notch is considered in detail.

A. Robinson (Jerusalem)

1101:

★Guderley, K. G. Theorie schallnaher Strömungen. Springer-Verlag, Berlin–Göttingen–Heidelberg, 1957. xv+376 pp. DM 42.00.

This book provides a thorough exposition and discussion, from a physicist's point of view, of so-called transonic flows. These are compressible inviscid flows in which there are regions where the speed of the fluid exceeds the speed of sound (supersonic) and other regions where it is less than the speed of sound (subsonic). The only cases for which there is much information are two-dimensional and axially symmetric flows.

The necessary fundamentals of inviscid compressible flow are developed in the first chapter along with an elementary approach to properly set boundary value problems for the simple model equations. Laplace's equation (for subsonic steady flow) the wave equation (for supersonic or time-dependent flow). These are also used to illustrate the types of perturbation that are used.

In chapter II, various basic topics and some formulae for transonic flows are developed: the Prandtl-Glauert approximation, shock conditions, boundary conditions and similarity laws. Chapter III contains various perturbation theories for almost sonic flow. In particular, one finds here a full treatment on the motion of an axially-symmetric body moved from rest to close to sonic velocity (Gardner-Ludloft).

However, all such perturbation theories break down essentially at the sonic velocity. The model equation for observing this breakdown in the steady case is $-\phi_x\phi_{xx}+\phi_{yy}=0$. Near sonic velocity ($\phi_x=0$), all solutions do not behave like solutions of $\phi_{yy}=0$. Chapter IV considers certain exact solutions of the nonlinear equation and examines the corresponding basic flows.

The two-dimensional case can often be solved because the equation for the potential as a function of the velocity (hodograph variables) is linear. Chapter VI introduces the properties of the hodograph transformation and chapters VI–X are concerned with two-dimensional flows treated by the hodograph method. In chapter VI, the reader will find extensive descriptions of various flows, e.g., wedge with shocks, Mach reflections, etc. Unfortunately the accompanying essential figures are so cramped that they are extremely hard to decipher.

Chapter VII is mainly devoted to a class of special solutions of the Tricomi equation (a good approximation to the potential equation in the hodograph plane). Various singular solutions can be constructed this way and the author has used these solutions, powerfully if somewhat laboriously, to construct special flows. Many examples of this work are in chapters VIII–X along with numerical results; in particular the construction of flows that are sonic at infinity is shown. These special solutions constitute a very strong technique for looking at local transonic situations. The mathematical properties developed here at length have been and will be extremely useful for this purpose.

Chapter XI turns to steady axially symmetric flows. Solutions are constructed as special solutions of a certain perturbation problem. The equations are not simplified in the hodograph plane. Thus the results are limited in comparison with those of two-dimensional flows.

This book is for the specialist who has good physical intuition. The student of equations of mixed type will find many open problems to study that are treated here only in special cases. The fluid dynamicist will be particularly glad to have many of Guderley's major contributions to the subject gathered in one place.

C. S. Morawetz (New York, N.Y.)

1102:

Shirokov, M. F. Velocity and temperature discontinuities near the walls of a body around which rarefied gases flow with transonic velocities. Soviet Physics. JETP **34** (7) (1958), 1029–1032 (1490–1495 Ž. Eksper. Teoret. Fiz.).

The regime considered is defined by $Ml/L<1$ where M is the Mach number, l the mean free path, and L the characteristic dimension of the flow. In this regime the gas-dynamic equations retain their usual form but there are discontinuities (e.g. slip) at solid boundaries. The velocity distribution function is determined by this statement, the condition of thermodynamic equilibrium, and Stokes' hypothesis. Conservation conditions applied at the boundary then lead to expressions for the velocity and temperature discontinuities in terms of an accommodation coefficient. The slip velocity is not changed in comparison with the low-speed results. The value of the temperature discontinuity is affected by the Mach number; approximately (and in special cases, exactly) the Maxwell boundary condition applies if the static temperature is replaced by the stagnation temperature.

W. R. Sears (Ithaca, N.Y.)

1103:

Kopylov, G. N. On similarity of transonic plane flows. J. Appl. Math. Mech. **22** (1958), 540–548 (391–395 Prikl. Mat. Meh.).

Les équations régissant les écoulements transsoniques faiblement perturbés peuvent s'écrire avec des notations qu'il est inutile de détailler

$$(K+U)\frac{\partial U}{\partial X}-\frac{\partial V}{\partial Y}=0,\quad \frac{\partial U}{\partial Y}+\frac{\partial V}{\partial X}=0,$$

si

$$K\varepsilon=1-M_\infty^2,\quad v_x=(1-\varepsilon AU)v_\infty,\quad v_y=\varepsilon^{3/2}AVv_\infty$$
$$y=l\varepsilon^{-1/2}Y,\quad x=lX,$$

ε étant un paramètre petit. Mais une certaine latitude subsiste dans la manière de relier A aux autres constantes de l'écoulement; on sait seulement que A^{-1} est équivalent à $(k+1)$ si k est l'indice adiabatique. La loi donnée par von Karman correspond précisément à $A^{-1}=k+1$; Spreiter a proposé $A^{-1}=(k+1)M_\infty^2$. L'auteur montre qu'il peut y avoir intérêt à choisir A suivant les propriétés du problème étudié. Par exemple, on approche au mieux les équations de la ligne sonique en prenant $A^{-1}=\frac{1}{2}+(k+\frac{1}{2})M_\infty^2$. Le choix de Spreiter est optimum pour approcher la polaire de choc. Plusieurs comparaisons avec des solutions connues illustrent la validité des divers choix possibles suivant les problèmes considérés.

P. Germain (Paris)

1104:

Bibosunov, I. An example of transonic flow of a gas with a supersonic zone, terminated by a curved shock which ends inside the flow. J. Appl. Math. Mech. **22** (1958), 431–441 (311–319 Prikl. Mat. Meh.).

L'auteur construit un exemple d'écoulement transsonique au voisinage d'un point où un choc rencontre une ligne sonique. Le problème est traité dans le plan de l'hodographe. L'exposant de la solution homogène de l'équation de Tricomi est choisi à priori sans justification. Toutes les conditions aux limites sont écrites avec les variables de l'hodographe, ce qui entraine des calculs assez longs, mais qui sont traités ici explicitement, et menés jusqu'aux valeurs numériques. L'auteur de cette analyse n'est pas certain toutefois que l'écoulement ainsi construit soit dépourvu de lignes limites.

P. Germain (Paris)

1105:

★Miles, John W. The potential theory of unsteady supersonic flow. Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge University Press, New York, 1959. xii+220 pp. (1 insert) $8.50.

This book, the latest in the series of Cambridge Monographs on Applied Mathematics, is a worthy addition to the brilliant, authoritative and indispensable volumes which have preceded it. The author, who in the fifty or more papers which he has written since the war has been the principal architect of the theoretical structure of the potential theory of unsteady supersonic flow, has provided, as no-one else could, a comprehensive organic and readable account of the present stage in its development.

Apart from an unfortunately but necessarily short chapter on linear problems and an odd reference to the scanty experimental results the whole of the book is devoted to the linearized potential equation of supersonic flow. The lack of experimental evidence is the most disquieting feature of the subject, especially in view of the sobering effect such evidence has had on the corresponding steady theory.

In the linearized potential theory of unsteady supersonic flow it is usually required to solve the four-dimensional wave equation

$$(M^2-1)\frac{\partial^2\varphi}{\partial x^2}+2\frac{M}{a_0}\frac{\partial^2\varphi}{\partial x\partial t}+\frac{1}{a_0^2}\frac{\partial^2\varphi}{\partial t^2}=\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2},$$

where φ is the potential, M (>1) the Mach number, a_0 the velocity of sound and x measures distance in the direction of the undisturbed stream. Although equations of this type have been studied for a long time and many special solutions and methods were available, which were indeed avidly seized on by the contributors to the subject, the determination of the lift and moment of the wings and bodies of importance in modern aerodynamics required new techniques and ideas to be developed. All these are described and appraised with lucidity and economy in this book. The only criticism of the book which occurs to your reviewer is that the author apparently feels the work of the aerodynamicist is done when he has found the lift and moment coefficients for the body in question. While this is strictly true it would be of interest to the student to see how the results he obtains are used in aeronautical engineering and it would further help to dispel any doubt that the subject was only of mathematical interest.

The chapter headings, which adequately describe the contents of the book, are: (I) The Linearized Equations; (II) The Transformation of the Subsonic Wing Equations; (III) The Transformation of the Supersonic Wing Equations; (IV) Reduction to Steady Flow; (V) Two-dimensional Problems; (VI) Simple Planforms; (VII) The Rectangular Wing; (VIII) The Quadrilateral Wing; (IX) Slender Wing Methods; (X) The Delta Wing; (XI) The Low Aspect Ratio Rectangular Wing; (XII) Slender Non-planar Bodies; (XIII) Non-linear Problems; Appendix, Reverse flow theorems.

The book is unreservedly recommended to all theoretical aeronautists.

K. Stewartson (Durham)

1106:

Marenkova, A. F. A thin wing in supersonic flow bounded by a wall. Vestnik Leningrad. Univ. **13** (1958), no. 19, 125–138. (Russian. English summary)

The paper deals with the problem of the supersonic flow around a wing placed near a plane or a slightly cambered wall. The author uses the linearized potential equation of gasdynamics, with its classical solution from the potential theory and the mirror reflection method which furnishes the proper boundary conditions. The technique is applied to two particular cases; a flat wing of an infinite span and a flat rectangular wing of a finite span. The solutions in both cases are obtained in elegant forms and the standard aerodynamic coefficients, like the lift coefficients, are represented as functions of the distance between the wing and the wall. A comparison of the theoretically calculated values with the wind tunnel tests performed at Mach number 2 and 3 shows a fair agreement.

M. Z. v. Krzywoblocki (Urbana, Ill.)

1107:

Lord, W. T.; and Brebner, G. G. Supersonic flow past slender pointed wings with "similar" cross sections at zero lift. Aero. Quart. **10** (1959), 79–102.

This is an application of slender-wing theory to the problem described by the title. The aim is to find wings, falling into this category, that have small wave drag and pressure distributions unlikely to be seriously affected by viscosity. Results are presented and discussed for various wings including those whose cross-sectional area distributions are described by sextic polynomials.

W. R. Sears (Ithaca, N.Y.)

1108:

Hirschfelder, Joseph O.; and Curtiss, Charles F. Theory of detonations. I. Irreversible unimolecular reaction. J. Chem. Phys. **28** (1958), 1130–1147.

The authors consider a one-dimensional steady-state continuous detonation wave in a simple gaseous system involving the exothermic chemical reaction $A \rightarrow B$. Effects of viscosity, diffusion, and thermal conductivity are included for Prandtl number $\frac{3}{4}$ and Lewis number 1. A system of first order ordinary differential equations (the customary energy and momentum integrals, etc.) determines gas velocity u, temperature θ, mole fraction x of A, and mass fraction G of A as functions of distance ξ. Since one integral of the system is known, x can be eliminated to reduce the order of the system to three. Conditions at the hot boundary correspond to a singular point at $\xi = \infty$. For a continuous wave the cold boundary should correspond to a singular point at $\xi = -\infty$. Strictly speaking, there is no such singular point, but for $x = G = 1$ the authors determine their cold boundary values θ_0 and u_0 from $du/d\xi = d\theta/d\xi = 0$, and $dG/d\xi$ turns out to be negligibly small. Linearized forms of the equations are used to describe behavior of solutions near hot and cold singularities. For various choices of detonation parameters the non-linear equations have been integrated numerically. The results do not agree with the von Neumann-Zeldovitch detonation model, i.e. shock followed by flame.

J. H. Giese (Aberdeen, Md.)

1109:

Linder, B.; Curtiss, C. F.; and Hirschfelder, J. O. Theory of detonations. II. Reversible unimolecular reaction. J. Chem. Phys. **28** (1958), 1147–1151.

The discussion and calculations of the second paper closely follow the pattern of the first.

J. H. Giese (Aberdeen, Md.)

1110:

Curtiss, C. F.; Hirschfelder, J. O.; and Barnett, M. P. Theory of detonations. III. Ignition temperature approximation. J. Chem. Phys. **30** (1959), 470–492.

In the first two papers the reaction rates for $A \rightarrow B$ and $B \rightarrow A$ depend exponentially on $-1/\theta$. In the third paper the temperature dependent factor in the rate of production of B is taken to be a step function which vanishes below a certain ignition temperature. Now $x(\xi)$ and $G(\xi)$ can be found explicitly, and θ can be eliminated by means of the integral used in the preceding papers. In the remaining first order differential equation for u the authors transform to independent variable x and integrate numerically. Solutions have been tabulated for many sets of detonation parameters. Best agreement with the results of the first paper is found when the ignition temperature exceeds the von Neumann shock temperature.

J. H. Giese (Aberdeen, Md.)

1111:

Kawai, Norio. On the acoustic field by a vibrating source arbitrarily distributed on a ribbon plate. I. J. Phys. Soc. Japan **13** (1958), 1374–1384.

"Two-dimensional rigorous solutions are presented for the acoustic field in air caused by a vibrating source arbitrarily distributed on an infinitely thin and infinitely long ribbon. Using rectangular coordinates, the unique solution satisfying the boundary conditions is obtained by the method of expansion in hypergeometric polynomials. The expression of the velocity potential at large distance, pressure on the ribbon plate and power of radiation are obtained as a function of $k = 2\pi a/\lambda$, where a is the half-breadth of the plate and λ is the wave length." (Author's summary)

L. B. Felsen (Brooklyn, N.Y.)

1112:

Biot, M. A. Reflection on a rough surface from an acoustic point source. J. Acoust. Soc. Amer. **29** (1957), 1193–1200.

The response of an acoustic point source above a rough surface represented by a distribution of small hemispheric protuberances on an infinite plane is analyzed on the assumption that the coherent scattering alone is of importance. An equivalent "average" boundary condition at the rough surface is derived and gives rise to an appropriate reflection coefficient in the Sommerfeld integral solution for a point source above a half space. The formal solution for the reflection due to the rough surface is then interpreted as arising from a concentrated image source in free space plus a line distribution of images extending from the specular image point to infinity with exponentially decreasing strength. [One should note in this connection some related earlier work by J. B. Keller, Math. Res. Group, New York Univ. Res. Rep. no. BR-2, 1953; MR **14**, 877]. Special cases of vanishingly small surface roughness, glancing incidence, and the effect of fluid viscosity are discussed in further detail.

L. B. Felsen (Brooklyn, N.Y.)

1113:

Pao, Yoh-Han. Hydrodynamic theory for the flow of a viscoelastic fluid. J. Appl. Phys. **28** (1957), 591–598.

A theory for the constitutive equations of viscoelastic liquids is obtained by considering small strains relative to a convected coordinate system, and linear viscoelastic relations in terms of discrete retardation or relaxation spectra for the strain components so defined. The strain rate is defined in terms of the usual velocity strain plus a term for convection and a term for rotation.

Steady state homogeneous simple shear flow is considered, the law being transformed to fixed axes, to give the steady state viscosity and steady-state elastic compliance. The former gives approximate agreement with experiment. Application to flow through pipes is also made. Generalization to three dimensions is briefly touched upon.

Reference is made to other analyses of this problem, and there is some disagreement. The reviewer failed to see how the strain components defined for the convected coordinate system would remain small, for example in continuous shear flow, as assumed in the analysis.

E. H. Lee (Providence, R.I.)

1114:

★Bhatnagar, P. L.; and Lakshmana Rao, S. K. Steady motion of non-Newtonian fluids in tubes. Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 225–234. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi+362 pp.

The authors examine again the problem already treated by A. E. Green and R. S. Rivlin [Quart. Appl. Math. **14** (1956), 299–308; MR **19**, 800] of the steady flow of non-Newtonian fluids through tubes. For a circular tube the inlet length required for steady flow to be established is calculated on the assumption that a parabolic flow pattern holds at the end of the inlet, and that the dissipation of energy required to change the flow pattern is equal to the change in kinetic energy over the inlet length. For non-circular tubes there occurs a secondary flow which is considered as a perturbation of the purely rectilinear flow, the equations governing this flow being solved in detail for an elliptical cross section. The analysis seems a little confused owing to the use of the same symbol for the cross viscosity coefficient and the flow function.

J. E. Adkins (Nottingham)

1115:

★Jain, M. K. On rotational instability in visco-elastic liquids. Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore December 24–27, 1957, pp. 217–224. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi+362 pp.

The stability of motion is examined for a non-Newtonian visco-elastic liquid moving between steadily rotating concentric circular cylinders. The properties of the liquid are specified by a linear relationship between stress, stress-rate and rate of deformation. By considering a small perturbation of the steady flow, dependent exponentially upon the time variable, the author obtains a characteristic value problem. The solution of this problem by a variational procedure gives a Taylor number which is about 8% higher than the classical value, and it is inferred that the presence of an elastic element in the liquid inhibits instability.

J. E. Adkins (Nottingham)

1116:

Pai, S. I. Shock wave propagation in an infinitely electrically conductive gas with transverse magnetic field and gravitation. Z. Angew. Math. Mech. **39** (1959), 40–49. (German, French and Russian summaries)

L'auteur étudie les mouvements rectilignes d'un fluide compressible doué d'une conductivité électrique infinie sous l'effet d'un champ magnétique et d'un champ de gravitation. Le fluide est mis en mouvement par un piston; au moyen de développements en séries, la trajectoire du choc et le mouvement du fluide sont déterminés. La présence d'un champ magnétique a pour effet d'augmenter la vitesse de propagation du choc et d'élargir ainsi le domaine influencé par l'onde de choc.

H. Cabannes (Marseille)

1117:

Akhiezer, A. I.; Liubarskii, G. Ia.; and Polovin, R. V. The stability of shock waves in magnetohydrodynamics. Soviet Physics. JETP **35** (8) (1959), 507–511 (731–737 Ž. Eksper. Teoret. Fiz.).

L'objet de cet article est l'étude de la stabilité des ondes de choc magnéto-aérodynamiques dans le cas des mouvements rectilignes. La méthode consiste à discuter les équations aux variations obtenues à partir des équations classiques de la magnéto-aérodynamique; lorsque les équations aux variations ne peuvent être résolues de façon unique, on en déduit que le choc initial dégénère en plusieurs ondes et cesse ainsi d'être stable. Parmi les conclusions obtenues, on peut noter que les ondes d'Alfven sont toujours stables et que lorsque le champ magnétique est parallèle à l'onde les ondes lentes sont instables, tandis que les ondes rapides sont stables.

H. Cabannes (Marseille)

1118:

★Gheorghiță, Șt. I. Capitole din teoria mișcărilor în mediile poroase. [Chapters in the theory of motion in porous media.] Academia Republicii Populare Romîne, Știință și Tehnică, No. 11. Editura Tehnică, Bucharest, 1957. 116 pp. (1 insert) Lei 3.25.

The present monograph constitutes an excellent analysis of most of the important aspects of fluid motion in porous media. The author presents rigorous solutions to a number of problems. Some of them are also of practical interest. The limitations of the different solutions are also indicated, whenever possible.

The mathematical treatment is primarily based on conformal mapping theory, and the method of separation of variables. A number of distinct physical configurations are studied in detail (e.g., rectangular, circular and elliptic cylindrical regions, etc.). Both two- and three-dimensional flow in homogeneous and non-homogeneous media are considered. The question of the appropriate boundary-conditions is treated carefully, and the limitations of the ensuing approximate analyses are also stated in some cases. Some infinite integrals involving the product of Bessel and exponential functions are evaluated. Reference is made to thirty-four different publications.

K. Bhagwandin (Oslo)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 772, 933, 934, 1116, 1186, 1199, 1210, 1245.

1119:

Hanke, W. Optische Umbildung des allgemeinen Vierecks in ein anderes. Optik **15** (1958), 219–224.

In a previous paper [Optik **12** (1955), 316–321; MR **17**, 71] the author treated the problem of imaging a general quadrilateral in space into a parallelogram or rectangle by means of an ideal optical system. The present paper extends the treatment to the imaging of a quadrilateral into a quadrilateral. *E. W. Marchand* (Rochester, N.Y.)

1120:

van Kampen, N. G. The method of stationary phase and the method of Fresnel zones. Physica **24** (1958), 437–444.

The asymptotic behaviour for large k, of integrals of the form

$$\iint_D g(x, y)e^{-ikf(x,y)}dxdy$$

(g, f real, D finite domain) is discussed qualitatively. Integrals of this type play an important role in the diffraction theory of optical aberrations. The relation between the method of stationary phase (first applied to this problem by the author in 1948) and the method of Fresnel zones is discussed, and reasons are given why in many physical problems the method of stationary phase should not be regarded as a special case of the method of steepest descent. *E. Wolf* (Rochester, N.Y.)

1121:

Pradhan, Trilochan. Plasma oscillations in a steady magnetic field: Circularly polarized electromagnetic modes. Phys. Rev. (2) **107** (1957), 1222–1227.

When propagating along the field, longitudinal, right polarized and left polarized plasma waves remain uncoupled. But a pure wave analysis leads to singular integrals owing to particles which encounter the polarized waves at cyclotron frequency. The difficulty is resolved by the methods which N. G. van Kampen [Physica **21** (1955), 949–963; MR **17**, 690] developed for longitudinal waves (unaffected by the field). No definite dispersion formula can be found for pure waves, since arbitrary additions to the principal value of the singular integral are possible.

The arbitrariness is removed by superposition of waves and fitting initial conditions. The moments

$$\iint (u_1 \pm iu_2)\delta f du_1 du_2$$

of the distribution perturbation δf are coupled, respectively, to the right and left polarized electric fields for all types of variation with time. Their initial values (functions of u_3), the fields (and their Hilbert transforms) determine the hitherto unknown additions to the principal values of the singular integrals. Fourier-decomposition of the waves at later times then leads to eventually dominant frequencies which are complex and represent ("Landau-") damping. Fortunately, polarized waves do not exhibit growth and Fourier analysis in time is justified. Laplace transformation, giving the same results, would appear to have been more natural and familiar for handling the initial value problem than singular integrals, principal values and Hilbert transforms.

For a Maxwellian longitudinal velocity distribution the required Hilbert transforms can be given in terms of error integrals or Kummer functions. The Appleton-Hartree formula is changed significantly by a temperature spread in the vicinity of the cyclotron frequency, in that the resonance of the refractive index is smoothed and damped in typical manner. The Kramers-Kronig dispersion relation is checked. {Apart from some minor errors and misprints, the omission of $(e/m)E \cdot V_u f_0$ in the basic equations 4 and 7, but included subsequently, might worry the uninitiated.} *O. Buneman* (Cambridge, England)

1122:

Bernstein, Ira B.; Greene, John M.; and Kruskal, Martin D. Exact non-linear plasma oscillations. Phys. Rev. (2) **108** (1957), 546–550.

The title word "oscillations" should be interpreted "non-uniformities": only undulatory, one-dimensional oscillations are dealt with and transformed to static distributions at the outset. Boltzmann's equation then states that the Boltzmann functions f_+, f_- depend on the total energy $E = \frac{1}{2}m_+v^2 + e\phi$, $\frac{1}{2}m_-v^2 - e\phi$ only. Given these functions, charge density and particle pressure become known functions of ϕ. Poisson's equation, or the constancy of total pressure (including the electrostatic pressure $(d\phi/dx)^2/8\pi$), determines the potential distribution $\phi(x)$. The pressure equation is a first order differential equation with, typically, undulatory solutions. {The particle pressure is not identified by the authors as such, and is denoted $V(\phi)/8\pi$: a confusing notation in view of the differing subsequent use of "V".}

Conversely, given $\phi(x)$, one determines $f_+(E)$ and $f_-(E)$ by solving an integral equation in the regions of trapping, the troughs of $\pm e\phi$. {The first term in the formula given for the trapped electron density seems to have the wrong sign, and the signs in front of $e\phi_{\min}$ in the integrand of the second term also seem wrong.} By alternately solving for trapped electrons and trapped ions, Boltzmann distributions can be fitted to almost any profile $\phi(x)$. Pulses and "shocks" can be constructed, propagating at arbitrary speed without dispersion.

Small deviations from uniformity are studied by suitable expansion. Taking $E = 0$ as the level of the rim of a trough, $f(E)$ differs from $f(0)$ like $(-E)^{\frac{1}{2}}$. {Again, the first term in the given formula seems wrong, while in the integrals $(V \pm e\phi_{\min})^{-\frac{1}{2}}$ should appear in place of $V^{-\frac{1}{2}}$.} Contrary to linearized theory, perturbations of f due to trapping go like the square root of potential perturbations. However, the width of the trap in velocity space is of the order of this same square root, and linear theory is salvaged if the trapped electrons are simulated by a delta-function $\delta(v)$. The authors check this simulation by comparing moments of the distributions. Agreement with N. G. van Kampen's version of the linear theory [Physica **21** (1955), 949–963; MR **17**, 690] is thus achieved. There is no Landau damping [L. D. Landau, Acad. Sci. USSR. J. Phys. **10** (1946), 25–34; MR **9** 401] of these perturbations, in conformity with Landau's theory if applied to singular initial distributions. Creation and stability of these persistent distributions remain uncertain.

O. Buneman (Cambridge, England)

1123:

Kelly, Don C.; Margenau, Henry; and Brown, Sanborn C. Cyclotron resonance. Method for determining collision cross sections for low-energy electrons. Phys. Rev. (2) **108** (1957), 1367–1371.

The Boltzmann equation for plasma electrons is solved assuming uniformity in space, a constant magnetic field, an alternating electric field at right angles to it, collisions with neutrals according to the Chapman-Cowling formula. The method of solving is that of H. Margenau [same Rev. **69** (1946), 508–513]: developing in spherical harmonics in velocity space, and suppressing higher harmonics of the impressed frequency in time. The isotropic part of the Boltzmann function is then constant, the first order spherical harmonics are purely sinusoidal: there is a kind of rotation of the distribution about the magnetic axis.

For velocity-independent collision frequency the isotropic part is Maxwellian, with the temperature raised by the applied field energy. For constant mean free path the results are similar though more complicated. The anisotropic parts can be deduced: R.F. currents and power absorption are calculated. In agreement with experiment, the cyclotron resonance half-width equals the collision frequency when the latter is constant and very similar resonance curves result when the mean free path is constant. *O. Buneman* (Stanford, Calif.)

1124:

Bohm, David; and Pines, David. A collective description of electron interactions. II. Collective vs individual particle aspects of the interactions. Phys. Rev. (2) **85** (1952), 338–353.

In der vorliegenden Arbeit, welche eine Fortsetzung einer früheren Veröffentlichung der selben Verfasser [dieselbe Rev. **82** (1951), 625–634; MR **12**, 886] ist, wird das Verhalten eines dichten Elektronengases unter Berücksichtigung nicht nur der transversalen elektromagnetischen Wechselwirkung der Elektronen (wie in der zitierten Arbeit) sondern auch der Coulombschen Wechselwirkung besprochen. Wegen der grossen Reichweite der Coulombschen Kräfte kann dieses Verhalten in zwei Teile zerlegt werden; die eine Komponente rührt von der kollektiven Schwingung des Systems her, das ist die sogenannte Plasmaschwingung, welche die Plasmafrequenz $\omega_p=(4\pi ne^2/m)^{\frac{1}{2}}$ (n bedeutet die Elektronendichte) besitzt. Die zweite Komponente wird dagegen von den noch übrigbleibenden Coulombschen Wechselwirkungen kurzer Reichweite verursacht und hängt mit der ungeordneten thermischen Bewegung der einzelnen Elektronen zusammen. Für Wellenlängen die grösser als die Debyesche charakteristische Länge λ_D ($\lambda_D^2=kT/4\pi ne^2$) sind, werden die Schwankungen grösstenteils durch die erste Komponente, für solche dagegen die kleiner sind, durch die individuellen Bewegungen der einzelnen Elektronen (zweite Komponente) beschrieben. Wenn sich ein Elektron mit einer grösseren Geschwindigkeit als der thermischen bewegt, so regt es die kollektive Schwingung des Systems an, was an dem Čerenkoveffekt erinnert.

T. Neugebauer (Budapest)

1125:

Bohm, David; and Pines, David. A collective description of electron interactions. III. Coulomb interactions in a degenerate electron gas. Phys. Rev. (2) **92** (1953), 609–625.

In dieser dritten Arbeit der Verfasser wird das Verhalten eines dichten (entarteten) Elektronengases unter Mitberücksichtigung der Coulombschen Wechselwirkung nach der Quantenmechanik berechnet. In zwei früheren Arbeiten [siehe obenstehend] wurde schon das selbe Problem behandelt, in der ersten auch nach der Quantenmechanik, jedoch ohne Berücksichtigung der longitudinalen Coulombschen Wechselwirkung, in der zweiten unter Mitberücksichtigung dieser Kraft, aber nur klassisch.

Den Ausgangspunkt der Berechnungen bildet die gewohnte quantenmechanische Hamiltonsche Funktion eines Systems von Elektronen unter Berücksichtigung der Coulombschen Kräfte. Zuerst wird diese Funktion so umgeformt, dass sie das von den Coulombschen Kräften grosser Reichweite herrührende kollektive Feld und die demzufolge auftretenden Plasmaschwingungen enthält, Ausserdem enthält dann die Hamiltonsche Funktion noch das Verhalten der einzelnen Elektronen, welche aufeinander und auf die kollektive Bewegung durch Kräfte kurzer Reichweite einwirken. Mit Hilfe einer weiteren kanonischen Transformation wird die Wechselwirkung zwischen Teilchen und Feld weitgehend eliminiert. Die erhaltenen Resultate sind zu denen in der zweiten Arbeit hergeleiteten ähnlich. {Die Anwendung der erhaltenen Resultate auf die Elektronentheorie der Metalle enthält eine weitere Arbeit von D. Pines [Phys. Rev. (2) **92** (1953), 626–636].} *T. Neugebauer* (Budapest)

1126:

Agranovich, V. M.; and Rukhadze, A. A. On the propagation of electromagnetic waves in a medium with appreciable spatial dispersion. Soviet Physics. JETP **35** (8) (1959), 685–686 (982–984 Ž. Eksper. Teoret. Fiz.).

A method for dealing with electromagnetic waves in a medium with spatial dispersion is presented which is more detailed than that given by Ginsburg. Expansions are obtained for the "direct" and "inverse" spatial dispersions. *Authors' summary*

1127:

Barron, D. W.; and Budden, K. G. The numerical solution of differential equations governing the reflexion of long radio waves from the ionosphere. III. Proc. Roy. Soc. London. Ser. A **249** (1959), 387–401.

"The previous papers in this series [K. G. Budden, same Proc. **227** (1955), 516–537; Philos. Trans. Roy. Soc. London. Ser. A **248** (1955), 45–72; MR **16**, 752; **17**, 434] have dealt with the differential equations governing the reflexion of long and very long radio waves from the ionosphere. In the present paper these equations are cast in a simpler form by using a generalized matrix admittance function **A**, and the corresponding equations are obtained in a form suitable for numerical integration by a step-by-step process from properly chosen initial solutions. From the value of **A** so obtained for a point below the ionosphere the reflexion coefficient matrix **R** is obtained, whose elements include the familiar reflexion coefficients. The equations are given in full for one case.

The equations, and the computational procedure, are checked by using them to calculate the reflexion coefficients of a sharply bounded homogeneous ionosphere and comparing the result with that calculated by other methods.

The new method of calculation is considerably faster than the old and will make possible calculation for

higher frequency waves where many more steps are needed in the numerical integration.

A theorem on the equivalence of certain directions of propagation is stated and proved." (From the authors' summary.)

The detailed results of this new calculation (performed on Edsacs 1, 2 at Cambridge Univ.) will be reported on in a subsequent paper. *E. Isaacson* (New York, N.Y.)

1128:

Berger, Martin J.; and Spencer, Lewis V. General relation between fluxes from collimated point and plane sources of radiation. Phys. Rev. (2) **113** (1959), 408–412.

The radiation density in a diffusing medium is discussed. Linear relations are shown to exist between the spatial moments of the radiation density due to a collimated point source and to oblique plane sources. The density distributions themselves are related by a Fourier-Hankel transformation. *H. W. Lewis* (Madison, Wis.)

1129:

Breitenberger, E. Theory of multiple scattering. Proc. Roy. Soc. London Ser. A **250** (1959), 514–523.

The rigorous theory of multiple scattering is formulated for monoenergetic particles normally incident on a foil of finite thickness. Energy loss in the foil is neglected. In the approximation in which backward-moving particles are neglected, the results are the same as those of Goudsmit and Saunderson, which are valid at all angles in an infinite medium, though not, of course, in a finite foil. The next approximation is complicated, but, in principle, susceptible to numerical evaluation.

H. W. Lewis (Madison, Wis.)

1130:

Levy, Bertram R.; and Keller, Joseph B. Diffraction by a smooth object. Comm. Pure Appl. Math. **12** (1959), 159–209.

The essential concepts of this paper have been described in the earlier articles by Seckler and Keller [J. Acoust. Soc. Amer. **31** (1958), 192–205, 206–216; MR **20** #6926, #6927]. In the first of these articles, as well as in part I of the present paper, geometrical optics approximations are derived while introducing, apart from the conventional incident, reflected and refracted rays, also "diffracted rays"; the latter consist partly of "surface rays" propagating along geodesics on curved material boundaries, partly of the tangents to surface rays at two individual points. The second of the above mentioned articles, as well as part II of the present paper, deal with asymptotic expressions (for large wave numbers) of the rigorous solutions of the problems under discussion; these expressions confirm the correctness of the assumptions necessary for the derivation of the changes of the field strength along the surface rays. However, the parameter fixing the relation between the amplitudes of an incident ray and a surface ray produced by it, and also the other parameter describing the field strength attenuation along a surface ray, then differ from mode to mode in the case of problems depending on a separation of variables.

The present paper differs from the earlier ones insofar as a possible inhomogeneity of space and the situation inside shadow regions (along "complex rays") have not been considered. On the other hand, the scalar treatment is supplemented here by a vector treatment based on Maxwell's equations, thus permitting applications to arbitrary electromagnetic fields. The cases of the diffraction of a plane incident wave by a cylinder of arbitrary cross-section and by a sphere are discussed in particular, mainly for the so-called impedance boundary condition. Other extensions with respect to the earlier articles concern: (a) A discussion of the field near caustics; the infinite field strength on caustics is avoided by multiplying the geometrical optics result by a correction factor which is determined by the ratio of a third-order and a second-order saddlepoint approximation for the corresponding rigorous solution; (b) the introduction of corrections depending on higher-order contributions with respect to the inverse wave number, these corrections being in accordance with an earlier paper by Keller, Lewis and Seckler [Comm. Pure Appl. Math. **9** (1956), 207–265; MR **18**, 43].

Unfortunately, some notations of this important paper may be confusing. The same symbol A is used for the geometrical optics amplitude as well as for Airy's diffraction integral. The understanding of table I on page 170 would have been clearer if the definition of its symbols had not been postponed until section 2 of part II.

H. Bremmer (Eindhoven)

1131:

Gosar, Peter. The propagation of light through an inhomogeneous medium with small differences of refractive indices. Slovenska Akad. Znan. Umet. Razred Mat. Fiz. Tehn. Vede. Ser. A **7** (1956), 37–116. (Text in Serbo-Croatian and English)

The author studies the effect of small variations of the refractive index on the propagation of light through non-homogeneous media. A number of different functions are employed to represent these variations.

The ensuing integrals, sometimes comprising Bessel functions, are in most cases evaluated by means of well-known techniques. The relation between the wave-length and the diameter of the scatterers is the principal criteria for the determination of the different regions of validity of the various approximations.

In particular, the author studies the Christiansen-filter, theoretically as well as experimentally. The theoretical results seem to agree with the experimental findings in some cases. The author presents some of his results in tabular form, as well as in figures (scattering cross-sections). In most cases, the calculations were carried out for the mathematically simpler expressions. However, it should be pointed out that there exist a good deal of data of recent investigations to which the author does not make any reference [cf. papers reviewed in MR concerning this topic]. *K. Bhagwandin* (Oslo)

1132:

Raggi, Bianca. Sull'instabilità dei modi TEM in un cavo multipolare. Boll. Un. Mat. Ital. (3) **13** (1958), 301–307. (English summary)

When the dielectric that fills a shielded two-wire transmission line becomes slightly inhomogeneous in the transverse coordinates, the original TEM mode of propagation splits into two approximately TEM modes whose phase velocities are unequal and independent of frequency.

C. H. Papas (Pasadena, Calif.)

1133:

Kacenelenbaum, B. Z. Critical cross-sections in non-uniform wave guides. Dokl. Akad. Nauk SSSR **123** (1958), 53–56. (Russian)

This paper is an extension of earlier work [Radiotekh. i Elektron. **2** (1957), 531–546]. The amplitude of parasitic waves is determined in a rectangular non-uniform slanted wave guide of arbitrary shape with ideal walls. It is found that a change of variables permits one to reduce this problem to the calculation of the coefficient of reflection from a non-uniform long line. *J. E. Rosenthal* (Passaic, N.J.)

1134:

Bryant, P. R. A topological investigation of network determinants. Proc. Inst. Elec. Engrs. C. **106** (1959), no. 9, 16–22.

Expressions are derived for general RLC networks to obtain the degree and power of the determinant polynomial in the complex-frequency variable λ. The nodal-admittance and loop-impedance matrices, as well as other network functions, are treated.

G. Kron (Schenectady, N.Y.)

1135:

Zemanian, Armen H. On transfer functions and transients. Quart. Appl. Math. **16** (1958), 273–294.

Let $Z(s)=KN(s)/D(s)$ be a system function, where $N(s)$ and $D(s)$ are polynomials of degree n and m with highest coefficient 1, K is real, $k=m-n>0$, and the zeroes of $D(s)$ are in the left half plane. The author defines subsets of these for each k which he calls the functions of class k and subclass k, the second set being a subset of the first. Several sufficient conditions are found for a $Z(s)$ to be in these sets. The second part of the paper derives properties of the class k and subclass k functions, such as bounds on the unit impulse response and unit step response, as well as restrictions on the rise time and settling time. *J. Blackman* (Syracuse, N.Y.)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 1052, 1053, 1054, 1055, 1062, 1087, 1088, 1108, 1109, 1110.

1136:

Evett, Arthur A. Alternate proof of the existence of the entropy function. Amer. J. Phys. **27** (1959), 361–363.

A proof is presented for the exactness of dQ/T based on showing directly that the reciprocity conditions are satisfied if one assumes the validity of Carnot's theorems.

Author's summary

1137:

Dmitrieva, M. I. Natural convection in a conical diffuser. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 3, 27–31. (Russian)

The paper deals with the problem of laminar, natural convection in a conical duct. The axis of the duct is vertical, the cone spreads upwards to infinity, and the mass flow at the vertex is different from zero. The walls of the cone are heated to a prescribed (not necessarily uniform) temperature and the fluid temperature at infinity is specified and constant. Alternatively, the heat flux across the cone wall is specified. The problem is solved by introducing a stream function and series expansions for the stream function, and the temperature and pressure distributions. The latter are in terms of reciprocal powers of the radial coordinate. Only the steady-state is considered. A general solution is obtained in terms of four linearly independent functions which are related to the Legendre functions of 1st and 2nd order.

Diagrams of typical velocity distributions for heating and cooling are given. These show clearly the occurrence of secondary flow characteristic of problems of this kind.

J. Kestin (Oxford)

1138:

Friedman, Avner. Generalized heat transfer between solids and gases under nonlinear boundary conditions. J. Math. Mech. **8** (1959), 161–183.

The following type of boundary value problem is treated: $L(u)=f(x,t)$ $(x\in B, t>0)$; $u(x,0)=\psi(x)$ $(x\in B)$; $\partial u/\partial T=g(x,t,u)$ $(x\in \dot{B}, t>0)$; where L is a second order parabolic differential operator with coefficients depending on x and t; B is a bounded domain in n-dimensional Euclidean space, with boundary $\dot{B}$, and

$$\left.\frac{\partial u}{\partial T}\right|_{(x,t)} = \lim_{y\to x} \sum a_{ij}(y,t)\cos(\nu, y_j)\frac{\partial u(y,t)}{\partial y_i},$$

with $y=(y_1,\cdots,y_n)$, $a_{ij}(y,t)$ the coefficient in L of $\partial^2 u/\partial y_i \partial y_j$, and (ν, y_j) the angle which the outer normal to B at x forms with the y_j-axis. Existence, uniqueness, and boundedness of solutions are established, and their behavior as $t\to\infty$ is investigated.

J. Elliott (New York, N.Y.)

1139:

Jäckel, Hans. Eine Methode zur Berechnung quasi-stationärer Temperaturzustände bei gesteuerten Wärmeleitvorgängen in festen Körpern. Z. Angew. Math. Phys. **10** (1959), 133–142. (English summary)

Let $T(P,t)$ denote the temperatures at points P in a solid at times t, where the boundary surface of the solid consists of r parts F_k $(k=1,2,\cdots,r)$ each having a high degree of regularity. The author considers the boundary value problem $T_t(P,t)=a^2\nabla^2 T(P,t)$, $T(P,0)=f(P)$, $dT/dn+\alpha_k T=\alpha_k\phi_k(P,t)$ when P is on F_k, where a^2 and α_k are positive constants. Under certain conditions on the prescribed functions f and ϕ_k he indicates why $T(P,t)$ tends to a well defined function $\bar{T}(P,t)$, called the quasi-stationary temperature, as t tends to infinity. By resolving T into components and using superposition and Duhamel formulas, he obtains a series representation of $\bar{T}$ in terms of eigenfunctions and eigenvalues of three-dimensional eigenvalue problems. An example is given for a circular cylinder with finite dimensions.

R. V. Churchill (Ann Arbor, Mich.)

1140:

Shen, S. F. The generalized thermal energy in boundary layer type flows with chemical reaction. Z. Angew. Math. Mech. **38** (1958), 431–436. (German, French and Russian summaries)

By introducing a "generalized thermal energy", which is defined as the maximum attainable enthalpy, i.e., the sum of the stagnation enthalpy and the potential heat release due to the combustible, some constant pressure combustion problems are mathematically simplified. The problem of the constant pressure ignition of a cold stream of combustible mixing with a hot stream of combustion products is re-examined. *L. N. Tao* (Chicago, Ill.)

QUANTUM MECHANICS

See also 694, 772, 978, 979, 1058, 1124, 1125.

1141:

★**McConnell, J. Quantum particle dynamics.** Series in Physics. North-Holland Publishing Co., Amsterdam; Interscience Publishers Inc., New York, 1958. xi+252 pp. $6.00.

Some 250 pages offer rather cramped accommodation for the development from introductory wave-mechanics to strange particle theory via quantum electrodynamics. Particularly in the earlier chapters, on Special Theory of Relativity, Introductory Quantum Theory, Wave and Matrix Mechanics, Angular Momentum, Many-particle Systems, the reviewer feels that a reader for whom these chapters are necessary will find parts of the treatment too facile to avoid perplexity. Examples are the operators of the Schrodinger representation (§ 10), the historical Heisenberg introduction of matrix multiplication (§ 14) and (p. 99) the use of x, y, z for the coordinates of a particle. The remainder of the book discusses the Dirac equation, Quantum Electrodynamics, photoelectric and Compton effects, Bremsstrahlung and pair production, field theory of nuclear forces, mesons, nucleons and strange particles. Here again the text may be occasionally criticised. The production of the book is good, the printing excellent. One can only regret that the author did not develop part or all of his subject matter much more fully.

C. Strachan (Aberdeen)

1142:

van Wyk, C. B. Charge conjugation and angular momentum. Nuovo Cimento (10) **6** (1957), 522–532. (Italian summary)

"The formal correspondence between spin and charge as implied by the idea of isobaric spin, is explained by showing that rotations in three- and four-dimensional spaces can be expressed as generalized phase transformations. Reflections in three dimensions are also expressed as phase transformations, the eigenfunctions of which are linearly related to those of suitable rotations. It is shown that the Kramers-Pauli charge conjugation is the charge analogue of spin in four dimensions. Hence the charge conjugation formalism presents a spin angular momentum framework which is applicable to particles with two types of spin. The four-dimensional orbital angular momentum eigenfunctions are given as a generalization of spherical harmonics." (Author's summary)

F. Rohrlich (Iowa City, Iowa)

1143:

Leipnik, Roy. Entropy and the uncertainty principle. Information and Control **2** (1959), 64–79.

In this paper the concept of entropy is generalized for quantum-mechanical systems in a pure state. Defining

$$L(\psi) = -\int |\psi(x) \log \psi(x)|^2 dx - \int |\phi(p) \log \phi(p)|^2 dp,$$

where $\psi(x)$ and $\phi(p)$ are the wave-functions in co-ordinate space and momentum space, respectively, it is shown that this 'joint entropy' has a lower bound, which is determined by a non-linear minimal condition. The result is interpreted in terms of the indeterminacy principle, and illustrated by its application to wave-packets. A further generalization for any pair of non-commutation observables is conjectured.

H. S. Green (Adelaide)

1144:

Stevens, K. W. H. The wave mechanical damped harmonic oscillator. Proc. Phys. Soc. **72** (1958), 1027–1036.

"The classical problem of a damped harmonic motion is formulated wave mechanically, with particular reference to the charge in a tuned circuit, and the electromagnetic field in a resonant cavity. Results in close agreement with the classical problem are obtained, without using perturbation theory. The classical frequency appears in the time dependence of the eigenfunctions and the classical damping appears as a decay in the eigenvalues. Zero point energy and thermal noise are discussed. The formulation is expected to break down, at very high frequencies, for tuned circuits and it is suggested that the same may be true for cavity oscillations and travelling electromagnetic waves." (Author's abstract)

E. T. Kornhauser (Providence, R.I.)

1145:

Foldy, L. L. Fermi-Segrè formula. Phys. Rev. (2) **111** (1958), 1093–1098.

The Fermi-Segrè formula with neglect of relativistic corrections can be written as

$$\psi_n^2(0) = \frac{Zz^2}{\pi a_0^3 \nu_n^3} \cdot \frac{d\nu_n}{dn} = \frac{Zz^2}{\pi a_0^3 \nu_n^3}\left[1 - \frac{d\Delta_n}{dn}\right].$$

Here $\psi_n^2(0)$ is the probability density for the S-electron at the nucleus in the nth stationary state, Z is the nuclear charge, z is the charge of the ion in the field of which the electron moves, $a_0 = \hbar^2/me^2$ is the Bohr radius, ν_n is the effective quantum number of the nth stationary state defined by the fact that the term value in Rydberg units is given by z^2/ν_n^2, and $\Delta_n = n - \nu_n$ is the quantum defect of the nth state. This formula is derived by making use of the effective-range method drawn from scattering theory and a refinement of the JBWK method which avoids problems with the Coulomb singularity of the ion core potential.

S. C. van Veen (Delft)

1146:

Shirokov, Iu. M. Relativistic theory of polarization effects. Soviet Physics. JETP **35** (8) (1959), 703–707 (1005–1012 Ž. Eksper. Teoret. Fiz.).

This paper extends the work of Blatt and Biedenharn [Rev. Mod. Phys. **24** (1952), 258–272] on the collision of polarized particles, to relativistic energies. The representations of the inhomogeneous Lorentz group are used instead of those of the rotation group. Only two body collisions are considered. The S matrix elements for the collision are specified by the values of a set of commuting operators which are either invariants of the Lorentz group or have a simple relativistic interpretation. These operators are the total energy (mass) and momentum, the total intrinsic angular momentum, a component of the intrinsic angular momentum, and the orbital angular momentum l, and spin angular momentum S in the centre of mass system. Except for l and S these quantities are also suitable for the case where the two particles in the initial or final states are interacting.

C. A. Hurst (Adelaide)

1147:

Agostinelli, Cataldo. Sulle equazioni di Schrodinger integrabili per separazione di variabili. Boll. Un. Mat. Ital. (3) **13** (1958), 11–23.

The problem of integrating the Schrödinger equation by separation of variables is reconsidered without the assumption that the coordinate system in which the separation takes place is orthogonal. [Previous work, e.g., L. P. Eisenhart, Ann. of Math. **35** (1934), 284–305; Phys. Rev. (2) **74** (1948), 87–89; MR **9**, 590, assumed the coordinates orthogonal.] The author shows that non-orthogonal systems are possible only when some of the coordinates are ignorable. He then shows how in the orthogonal case the problem of solving the Schrödinger equation can be reduced to finding solutions of Riccati's equation. *A. S. Wightman* (Princeton, N.J.)

1148:

Fain, V. M. Quantum phenomena in the radio spectrum. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **12** (1958), no. 4 (27), 47–94. (Romanian)

A translation from the Russian [Uspehi Fiz. Nauk **64** (1958), 273–313; MR **20** #2210].

1149:

Spence, R. D. Angular momentum in sphero-conal coordinates. Amer. J. Phys. **27** (1959), 329–335.

1150:

★Боголюбов, Н. Н.; Медведев, Б. В.; и Поливанов, М. К. Вопросы теории дисперсионных соотношений. [Bogolyubov, N. N.; Medvedev, B. V.; and Polivanov, M. K. Questions in the theory of dispersion relations.] Contemporary Problems of Mathematics. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 203 pp. 7 rubles.

In the last few years there has been developed in the quantum theory of fields a new and very remarkable method, called the theory of dispersion relations. The main object of this approach is to establish relationships between the hermitian part of the scattering amplitude and definite integrals taken over its antihermitian part with respect to energy. The fact that such a relationship depends, inter alia, on the causal character of the theory and that it gives relationships between directly measurable entities, endows the theory of dispersion relations with great importance both in the eyes of the theoretician and in the eyes of the experimentalist. Since this field of research is still in a phase of rapid development, there was hitherto no complete account of it in the form of a monograph. The present book, whose manuscript was completed early in 1957, can claim, therefore, particular attention. The authors themselves have contributed very essentially to the development of this field of research. The present book contains a detailed exposition of the mathematical structure of the theory, including questions related to generalized functions and to functions of several complex variables. Much of the mathematical work is contained in chapter 1 and appendix A. Chapter 2 deals with basic physical assumptions and principles; chapter 3 discusses the relationships between radiation operators; chapters 4 and 5 concern the vacuum averages of the boson- and fermion-radiation operators, respectively. Chapters 6 and 7 establish the dispersion relations; chapter 8 adapts the theory, developed hitherto in an abstract mathematical fashion, to definite cases of physical interest. *P. Roman* (Manchester)

1151:

Chadan, Kh. On the connection between the S-matrix and a class of non-local interactions. I. Nuovo Cimento (10) **10** (1958), 892–908. (Italian summary)

This is a generalization of the work of Gourdin and Martin [Nuovo Cimento (10) **6** (1957), 757–779; **8** (1958), 699–707; MR **20** #1544] on (non-relativistic) non-local separable interactions between two colliding particles, to the case of a superposition of such an interaction with a local one. Only central interactions are considered and only S-states. If the local interaction is assumed to be known, the phase-shift and the bound state energies determine the non-local interaction provided the phase-shift is well behaved. The method of proof is analogous to that given by Gourdin and Martin.

F. Rohrlich (Iowa City, Iowa)

1152:

Gasiorowicz, S.; and Noyes, H. P. Dispersion relation for potential scattering. Nuovo Cimento (10) **10** (1958), 78–89. (Italian summary)

A dispersion relation is derived for the scattering amplitude of a Klein-Gordon particle, scattered by a static potential, using the causality condition, and field-theoretic methods. The results are similar to those obtained by Khuri, using an explicit solution for the Schrödinger equation, and no causality requirement. It is necessary, however, to add to causality a statement about the asymptotic behavior of the scattering amplitude, and this is taken from Khuri's work. It remains true that the potential must fall off at least exponentially for any range of validity of the dispersion relation.

H. W. Lewis (Madison, Wis.)

1153:

Logunov, A. A.; and Isaev, P. S. On the theory of dispersion relations for photon-nucleon scattering. Nuovo Cimento (10) **10** (1958), 917–942. (Italian summary)

The proof of the dispersion relations for forward scattering is based on the local character of the meson- or the nucleon-fields. For an arbitrary scattering angle, a deeper analysis is required, based on the local character of both the meson- and the nucleon-fields. In the present paper the exact proof is given that the relations derived previously for the Compton effect on nucleons in absence of an unobservable region are valid for an arbitrary scattering angle. The method of investigation is that suggested by Bogoljubov [N. N. Bogoljubov, *Report at the international conference on theoretical physics*, Seattle, September 1956]. Electromagnetic corrections are not taken into account. *P. Roman* (Manchester)

1154:

Logunov, A. A.; and Tavkhelidze, A. N. Generalized dispersion relations. Nuovo Cimento (10) **10** (1958), 943–952. (Italian summary)

The method for obtaining dispersion relations for the reactions of the type $a+b \to a'+c+d$ (with one nucleon and one boson in the initial state and with one nucleon and two bosons in the final state) are given, without additional assumptions about the equality of energy of the final bosons. First the kinematics of the process are studied, then the antihermitian part of the amplitude is analyzed, finally the dispersion relations are deduced.

P. Roman (Manchester)

1155:

Logunov, A. A.; Bilenkij, S. M.; and Tavkhelidze, A. N. On the theory of dispersion relations. Nuovo Cimento (10) **10** (1958), 953–964. (Italian summary)

The proof of the dispersion relations for double Compton scattering $\gamma+p\to 2\gamma+p$ in the case of absence of an unobservable energy region is given. First of all the retarded and advanced amplitudes are constructed with the help of the causality principle. These functions are determined for real energy values above the threshold of the process. Then two functions Φ^r and Φ^a are constructed from the amplitudes, which are analytic in the upper and lower energy-half-planes respectively, and determine a function Φ which is analytic in the whole plane, except for the cut on the real axis. Dispersion relations arise then by applying Cauchy's theorem to this function.

P. Roman (Manchester)

1156:

★Rzewuski, Jan. Field theory. Part I: Classical theory. Polish Academy of Sciences, Physical Monographs. Państwowe Wydawnictwo Naukowe, Warsaw, 1958. 297 pp.

This monograph is part I of a two-part work, the second part to be concerned with quantum field theory. Part I is concerned with both local and non-local Lorentz-covariant theories whose field equations can be derived from variational principles. Non-local field theories are described as field theories of which the action principles are integrals of integrands depending on the field variables at N different space-time points, and hence extended N-fold over the domain of definition. If $N=1$, the theory is said to be local. The author explores both general theorems applying to this whole class of field theories, and special applications.

There is a very detailed table of contents but no index, which is probably intended to complete part II. The topics treated are the following: Ch. I, Spinors and tensors; ch. II, Variational principles (this chapter includes a discussion of the conservation laws following from Lorentz invariance and from gauge invariance of the first kind, and also a discussion of the variational principles of relativistic mechanics); ch. III, Examples (these range from the cases of the various non-interacting fields to the non-quantum treatment of the mutually interacting Maxwell and Dirac fields and include Fokker's theory of mutually interacting classical point electrons, non-local theories of the Kristensen-Møller type, and Born and Infeld's non-linear theory of the electromagnetic field); and ch. IV, the Basic solutions of the field equations (detailed discussion of the Δ- and S-functions of relativistic field theory).

The author's treatment of the subject matter is broad and in some instances even slow. For reading the book no previous knowledge of the field-theoretical literature is required; some results of the classical theory will be found here that are otherwise fairly inaccessible. The notation appears somewhat ponderous, but this impression results undoubtedly in large part from the author's program to treat local and non-local theories in a unified manner, and by means of a formalism that is ab initio applicable to both. Reading part I certainly whets one's curiosity for part II.

P. G. Bergmann (Syracuse)

1157:

★Thirring, Walter E. Principles of quantum electrodynamics. Translated from the German by J. Bernstein. With corrections and additions by Walter E. Thirring. Pure and Applied Physics. Vol. 3. Academic Press Inc., New York-London, 1958. xv+234 pp. $8.00.

This is not simply a translation of the German edition [Franz Deuticke, Vienna, 1955; MR **18**, 972]. The book has also been largely rewritten and reorganized, devoting more space to recent developments. It may well be called a new edition and constitutes in many respects a considerable and successful improvement over the original version. The organization is more systematic and pedagogically more desirable. After the general introduction (part I) the reader is led from the free fields (part II) via an illuminating discussion of a single quantized field with external sources (part III) to the interaction of the two quantized fields, the photon and the electron field (part IV). The bridge offered by part III permits an exposition of such topics as infrared divergence, vacuum polarization, and charge renormalization before confronting the reader with the much more difficult problem of interacting fields. In the treatment of this latter problem one finds the important section on renormalization theory completely rewritten and enlarged four-fold over the German edition. This theory is presented in terms of Lehmann's spectral functions. It is clear and readable, despite an oversimplification in connection with electron propagators, and it permits an intelligible discussion of Källén's work in the final section. In the section on higher order corrections the perturbation calculations are preceded by a brief exposition of Källén's renormalized equations. The concluding section is a short but well-rounded summary of the various investigations on the limits of quantum electrodynamics, including the work by Källén, Landau, Lee, and others. The illustrations are much improved over the German edition and so is the technical printing. The addition of a subject index is very helpful. All these improvements contribute to making this book an even better introduction into this difficult subject matter.

F. Rohrlich (Iowa City, Iowa)

1158:

Ter-Mikaelyan, M. L. Quantum electrodynamics in a medium. I. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **11** (1958), no. 4, 13–19. (Russian. Armenian summary)

1159:

Marx, G. The fundamental theorem of continuous transformations in the quantum theory. Acta Phys. Acad. Sci. Hungar. **9** (1958/59), 393–402.

By regarding the field equations, the Lagrangian, and the commutation rules as given, it is shown that the unitary operator associated with a symmetry transformation of the theory is determined. A construction is given for the generator of the transformation.

H. W. Lewis (Madison, Wis.)

1160:

Ferrari, E.; and Jona-Lasinio, G. On the causal propagation function of a Dirac field. Nuovo Cimento (10) **10** (1958), 310–317. (Italian summary)

It is shown that the causal propagation function of a Dirac field can be split in two functions which satisfy a

system of two integral equations similar to the couple of Low equations for the symmetrical charged fixed-source meson theory. (From the authors' summary)

G. Källén (Lund)

1161:

Caianiello, E. R. Perturbative expansions, field equations and their renormalization. Nuovo Cimento (10) **9** (1958), supplemento, 569–587.

An expository paper which presents primarily a summary of the author's previous work on this subject, and is the text of his talk delivered at the Colloque sur les Problèmes Mathematiques de la Théorie Quantique des Champs, Lille, June 1957.

F. Rohrlich (Iowa City, Iowa)

1162:

Okabayashi, T. On the causal character in field theory. Nuovo Cimento (10) **9** (1958), supplemento, 599–625.

This paper is a general survey of the role of causality in present day quantum field theory. The author begins by reviewing the Stueckelberg and Rivier results on causality [Helv. Phys. Acta **23** (1950), 215–222; MR **11**, 569] which show that the "causal" Green's function is determined for time-like values of the argument. However, the causality condition leaves the Green's function arbitrary for space-like regions. It is then shown that the "outgoing wave" Green's function (i.e. the vacuum expectation value of the time ordered product of two field operators) satisfies the Stueckelberg causality conditions provided the eigenspectrum of the field Hamiltonian is bounded from below. Hence the S-matrix [as defined by Lehmann, Symanzik and Zimmermann, Nuovo Cimento (10) **1** (1955), 205–225; MR **17**, 219] is causal when this condition is satisfied.

The author next discusses the relation between the causally undetermined part of the Green's function and the renormalizations in quantum field theory [E. C. G. Stueckelberg and D. Rivier, Helv. Phys. Acta **23** (1950), 236–239; MR **11**, 763; E. C. G. Stueckelberg and T. A. Green, Helv. Phys. Acta **24** (1951), 153–174; MR **13**, 191]. Since the renormalization constants are formally infinite, the author tentatively suggests that modifications of present theory should be made in this causally undetermined space-like region.

It is shown that the condition of "microscopic" causality (i.e. the requirement that two Bose type operators commute when their arguments are separated by a space-like distance) is contained in the Stueckelberg definition of causality. However, the author points out that the derivation of the dispersion relations really requires only an analyticity condition on the scattering amplitude which may hold even in a non-causal theory. Thus, the validity of the dispersion relations do not guarantee the causal nature of the theory.

The paper concludes with two appendices dealing with some remarks on non-local field theories and local field theories with higher derivatives.

R. Arnowitt (Syracuse, N.Y.)

1163:

Parasyuk, O. S.; Petrina, D. Ya.; and Tacunyak, P. I. The Čellen-Leman theorem in a space with indefinite metric. Ukrain. Mat. Ž. **10** (1958), 344–346. (Russian)

At the present time the quantum theory of fields in the Hilbert space with an indefinite metric is subject to numerous investigations. Simultaneously, some writers develop the theory of operators in the Hilbert space with an indefinite metric. In that respect the authors of the present paper call attention to works of L. S. Pontrjagin [Izv. Akad. Nauk SSSR Ser. Mat. **8** (1944), 243–280; MR **6**, 273], generalized to some extent by M. G. Kreĭn and others. In the present note the authors point out certain relations between these two approaches. In particular, they demonstrate how one can obtain the analogy of the Čellen-Leman theorem by the method proposed by Kreĭn. Starting from a convergent series in a Hilbert space they apply an operator equation which actually represents the equation of motion of quantum mechanics in Heisenberg's sense. After certain operations of the nature proposed by I. S. Iohvidov and M. G. Kreĭn [Trudy Moskov. Mat. Obšč. **5** (1956), 367–432; MR **18**, 320], they arrive at a system of two equations interpreted as an analogy to the Čellen-Leman theorem.

M. Z. v. Krzywoblocki (Urbana, Ill.)

1164:

Pais, A.; and Serber, R. A general transformation of the symmetrical pseudoscalar theory. Phys. Rev. (2) **113** (1959), 955–958.

The symmetrical pseudoscalar theory with fixed extended source is treated, and a rigorous transformation is accomplished. The Hamiltonian then depends explicitly on the total isotopic spin and total angular momentum of the system. Though the formalism is similar to that used in strong-coupling theory, no restriction to strong coupling is made. No applications are made of the formalism.

H. W. Lewis (Madison, Wis.)

1165:

Zyrianov, P. S. Generalized self-consistent field equations. Soviet Physics. JETP **35** (8) (1959), 309–311 (448–451 Ž. Eksper. Teoret. Fiz.).

A modification is made of the Hartree equations, to take into account scattering at small distances, and on this basis dispersion laws for the normal modes of Bose and Fermi systems are derived. In the limit in which these equations go over into the Hartree-Fock equations, the results are related to the known results.

H. W. Lewis (Madison, Wis.)

1166:

Goshen, S. [Goldstein, S.]; and Lipkin, H. J. A simple independent-particle system having collective properties. Ann. Physics **6** (1959), 301–309.

The partial successes of the shell model and the collective model of nuclear structure have led to attempts to formulate these two procedures as alternative descriptions of a single system. In a recent paper J. P. Elliott [Proc. Roy. Soc. London. Ser. A **245** (1958), 562–581; MR **20** #697] has shown that by taking linear combinations of degenerate energy levels in the shell model one can construct states having properties characteristic of the collective model. The authors proceed differently, making use of a re-formulation of the Hamiltonian operator in terms of dynamical variables which are to describe a collective type motion. In their procedure the collective states are not restricted to be linear combinations of

degenerate energy levels of the individual particle problem. The authors' method is exhibited explicitly only for the case of a linear oscillator potential field, and is based on the construction of a 3-parameter group of transformations of Lorentz type whose Lie operators are used for the definition of the collective coordinates.

E. L. Hill (Minneapolis, Minn.)

1167:

Goshen, S. [Goldstein, S.]; and Lipkin, H. J. A simple model of a system possessing rotational states. Ann. Physics **6** (1959), 310–318.

The collective model of nuclear structure makes use of the concept of sets of energy levels having the structure of rotational states. While groups of states having the required energy structure have been identified empirically, the appropriate geometrical parameters, such as the moments of inertia, which are inferred from the empirical data do not always seem to be realistic in terms of expected nuclear properties. The authors investigate the question whether an energy level structure of the rotational type actually implies rotational motion of a quasi-rigid type. The latter concept is defined in terms of the possibility of separation of the variables in a dynamical problem by means of geometrical coordinates representing a rotating system. The argument is carried out by consideration of the specific example of a 2-dimensional linear oscillator with an additional member proportional to the square of the angular momentum in the Hamiltonian function. This leads to an energy level structure of the desired rotational type. However, the separation of the rotational and vibrational motions involves an interaction between the two types of motion. It is concluded that the concept of rotational structure of the energy levels is not necessarily the same as that of simple geometrical rotation in space. The implications of the theory for nuclear structure theory are discussed.

E. L. Hill (Minneapolis, Minn.)

1168:

Clinton, William L.; and Rice, Bernard. Reformulation of the Jahn-Teller theorem. J. Chem. Phys. **30** (1959), 542–546.

The electrostatic theorem of Hellmann and Feyman is employed in a reformulation of the theory of the Jahn-Teller effect. This approach provides a much simpler physical picture of the phenomenon than the usual discussion in terms of the total energy, but is less suitable for numerical calculations. The general theory is illustrated by the case of the borane molecule (BH_3).

A. C. Hurley (Melbourne)

1169:

Datta Majumdar, S. Energy levels of triatomic molecules. Proc. Phys. Soc. Sect. A **67** (1954), 351–364.

"The mathematical theory of the asymmetrical top and of triatomic molecules has been considerably simplified by removing two of the Eulerian angles from the wave equation by a special artifice. Asymptotic expressions for the eigenvalues of the former and the centrifugal and Coriolis corrections to be applied to the energy levels of the latter have been obtained in closed forms for values of J exceeding 9 without solving any secular equation." (Author's abstract)

A. C. Hurley (Melbourne)

1170:

Maradudin, A. A.; and Weiss, George H. A method for evaluating lattice sums. Canad. J. Phys. **37** (1959), 170–173.

"A rapidly convergent method for evaluating a certain class of lattice sums by the use of asymptotic series is presented, and the method is illustrated by applying it to the evaluation of lattice sums previously studied by Benson and Schreiber." (Authors' abstract)

A. C. Hurley (Melbourne)

1171:

Amat, Gilbert; et Henry, Lucien. Sur les coefficients d'interaction entre la vibration et la rotation dans les molécules polyatomiques. I. Cahiers de Phys. **12** (1958), 273–286. (English summary)

"Orthogonality relations are established between the interaction coefficients $a^{\sigma\delta}_{ss'}$, $A^{\sigma\delta}_{ss's''}$ $\xi^{\alpha}_{ss's''}$. They enable one to compute more easily these coefficients and show that the rotation-vibration Hamiltonian does not depend upon the $A^{\sigma\delta}_{ss's''}$ coefficients. Rotation-vibration interaction coefficients of linear molecules are studied in the second part of the paper." (Authors' summary)

A. C. Hurley (Melbourne)

1172:

Landau, L. On the conservation laws for weak interactions. Nuclear Phys. **3** (1957), 127–131.

This is one of three classic papers logically consequent upon the suggestion of non-conservation of parity in weak interactions, the others being T. D. Lee and C. N. Yang, Phys. Rev. **105**, 1671 (1957) and A. Salam, Nuovo Cimento **5**, 299 (1957), both of which are reviewed below. This paper was apparently constructed almost immediately after the author saw the paper of Lee and Yang [T. D. Lee and C. N. Yang, Phys. Rev. (2) **104** (1956), 254–258] with its exhaustive suggestions for experiments to test the non-conservation hypothesis. From the discussion of the $\tau-\theta$ puzzle, the inference of non-conservation of parity is drawn explicitly and the suggestion made that the law of combined inversion—the product of space inversion and charge conjugation—be retained as a symmetry. It is then recognized that the non-conversion of parity can be ascribed to the neutrino, and the latter represented by a two-component spinor equation. The consequences of this assumption are spelled out most completely for the process of μ decay. This short paper contains no derivations, but rather consists entirely of the statement of hypotheses and results.

A. Klein (Philadelphia, Pa.)

1173:

Salam, Abdus. On parity conservation and neutrino mass. Nuovo Cimento (10) **5** (1957), 299–301.

This brief note was written shortly after Yang and Lee made their famous suggestion that parity might not be conserved in the weak interactions [C. N. Yang and T. D. Lee, Phys. Rev. (2) **104** (1956), 254–258]. The author points out the possibility that the parity violation in neutrino decays may be related to the vanishing of neutrino mass and self-mass. Both are to be secured by requiring that the Lagrangian of the theory be invariant under the substitution $\psi\rightarrow\gamma_5\psi$. It is pointed out that this leads to zero magnetic moment for the neutrino and strong restrictions on the shape of the μ decay spectrum.

The above suggestion does not lead to a natural explanation of parity non-conservation in decays not involving neutrinos. On that subject the author makes proposals too complicated to summarize here. [For similar independent suggestions of γ_5 invariance, see T. D. Lee and C. N. Yang, Phys. Rev. (2) **105** (1957), 1671–1675, reviewed below; and L. Landau, Nuclear Physics **3** (1957), 127–131, reviewed above].

A. S. Wightman (Princeton, N.J.)

1174:

Lee, T. D.; and Yang, C. N. Parity nonconservation and a two-component theory of the neutrino. Phys. Rev. (2) **105** (1957), 1671–1675.

At the time of reviewing, the present paper is already a classic in the field of elementary particle theory. Together with papers of Landau [#1172 above] and of Salam [#1173 above], it contained the first suggestion that a non-conservation of parity in weak interactions could be interpreted in terms of the properties of the neutrino, by replacing the Dirac equation for the latter by a two-component wave equation invariant under proper Lorentz transformations. The paper contains some still useful remarks on the adaptation of the familiar calculus of the Dirac equation to the two component form. Some of the consequences of the two component theory are also worked out, but these have been either specialized or altered in the currently accepted version of the theory, the so-called V-A form. *A. Klein* (Philadelphia, Pa.)

1175:

Nishijima, K. On the theory of leptons. Nuovo Cimento (10) **5** (1957), 1349–1354.

The author proposes an alternative form of γ_5 invariance [for the usual form see the three preceding reviews of: A. Salam, Nuovo Cimento (10) **5** (1957), 299–301; T. D. Lee and C. N. Yang, Phys. Rev. (2) **105** (1957) 1671–1675; L. Landau, Nuclear Physics **3** (1957) 127–131] of physical interactions in which $\psi \to \exp i\alpha\gamma_5\psi$ and $\bar{\psi} \to \bar{\psi} \exp i\alpha\gamma_5$. He shows that the S-matrices arising from the theories with field equations

$$[\gamma_\mu(\partial_\mu - ieA_\mu) + m]\psi = 0, \quad \Box A_\mu = -ie\bar{\psi}\gamma_\mu\psi;$$

$$[\gamma_\mu(\partial_\mu - ieA_\mu) + m \exp 2i\alpha\gamma_5]\psi = 0, \quad \Box A_\mu = -ie\bar{\psi}\gamma_\mu\psi$$

(which are obtained from one another by the above transformation) are the same. He investigates the circumstances in which this invariance should have the same consequences for certain weak interactions as the usual γ_5 invariance, with results too complicated to quote here. A novel feature of the paper is that invariance is assumed for the S-matrix rather than for the equations of motion themselves. *A. S. Wightman* (Princeton, N.J.)

1176:

Blin-Stoyle, R. J. On the universal Fermi interaction. Nuovo Cimento (10) **10** (1958), 132–135. (Italian summary)

The mesonic renormalization correction to the axial vector β-decay coupling constant is calculated in the lowest order and in the static limit. This shows the incorrectness of a suggestion by this reviewer [Nuovo Cimento (10) **8** (1958), 179–180, letter to the editor] that this correction would vanish if the nucleons were coupled to an isoscalar meson in such a way that the axial vector current was conserved. The reason suggested for this is that the effective part of the axial vector current is related to the three spatial components rather than the time component as is the case for the vector current.

Various ways of making the renormalization effect an enhancement are discussed.

J. C. Polkinghorne (Cambridge, England)

1177:

van Wyk, C. B. Covariant description of spin. Nuovo Cimento (10) **10** (1958), 854–862. (Italian summary)

For a particle of mass 1 one defines the four mutually orthogonal unit-vectors p^μ, s^μ, m^μ, and n^μ, where p^μ is time-like, the others space-like. The angular momentum tensor L ($L_{\mu\nu} = -i(m_\mu n_\nu - m_\nu n_\mu)$) and its dual L^D ($L_{\mu\nu}{}^D = p_\mu s_\nu - p_\nu s_\mu$) are the infinitesimal generators of the most general proper single-valued Lorentz transformation containing six independent parameters and two real variables ϑ and φ: $X_\mu' = \exp[iL\vartheta + L^D\varphi]x_\mu$. When considered in the rest system this transformation permits the interpretation of p^μ and s^μ as energy-momentum and spin vectors respectively, such that $s_\mu = L_{\mu\nu}{}^D p^\nu$. For mass-less particles p^μ and s^μ are null-vectors, $L_{\mu\nu}{}^D = 0$ and $s^\mu = \lambda p^\mu$.

F. Rohrlich (Iowa City, Iowa)

1178:

Schremp, E. J. G-conjugation and the group-space of the proper Lorentz group. Phys. Rev. (2) **113** (1959), 936–943.

Basing his arguments on considerations concerning the group manifold of the proper Lorentz group, the author has previously suggested a theory in which both spin and isotopic spin appear intrinsically in the theory of spin $\frac{1}{2}$ particles [Phys. Rev. (2) **85** (1952), 721; **99** (1955), 1603; **108** (1957), 1076–1077; MR **17**, 114; **19**, 1231]. In the present paper it is indicated that this involves a certain definite conception of the spatiotemporal character of "internal" elementary particle properties such as isotopic spin. It is found that the particular spatiotemporal character of the 3-parameter group of isotopic spin rotations of a strongly interacting charge doublet is identical with that of a group of spinor transformations which have been applied to the neutrino by Pauli [Nuovo Cimento (10) **6** (1957), 204–215; MR **19**, 612]. The author concludes that there should exist an analogue of isotopic spin rotations for leptons in general (although with a different physical interpretation). A criterion for the classification of all spin $\frac{1}{2}$ particles into the lepton and baryon families is suggested on the basis of a generalisation of the notion of G-conjugation. A unified quaternion formalism is used effectively in this paper.

H. Rund (Durban)

1179:

Fedorov, F. I. Projection operators in the theory of elementary particles. Soviet Physics. JETP **35** (8) (1959), 339–344 (493–500 Ž. Eksper. Teoret. Fiz.).

In the solution of various problems about particles with spin and their interactions, it is necessary to know the wave functions characterizing a particle that has given energy and momentum and is in a prescribed mass- and spin-state. Usually these wave functions are found by solving the free-field equation

$$(\gamma_k\nabla_k + \kappa)\psi = 0,$$

where γ_k are square matrices, with a definite choice of the basis for the representation in the space of the functions ψ. For the classification into the various spin-states one must further solve the equations for the characteristic values of the spin operator. Not only are these calculations very cumbersome, but also this method is in principle unsatisfactory because the physically significant results must be invariant under a change of the basis. It is shown that the use of minimal polynomials [defined in F. Fedorov, Dokl. Akad. Nauk SSSR **79** (1951), 787–790; MR **13**, 410; and in the present paper] makes it possible in the general case to reduce the calculation of the main physical quantities to the computation of the traces of certain combinations of matrices. The minimal polynomials are used to construct projection matrices which are dyads describing arbitrary possible states, and then the fundamental physical quantities are expressed directly in terms of these projection operators in an invariant way. As examples, the general conditions for definiteness of the energy and charge are obtained, and a simple derivation of the general commutation relations is given.

P. Roman (Manchester)

1180:

Novožilov, Yu. V.; and Prohorov, L. V. On the nucleon-hyperon forces from the meson theory. Vestnik Leningrad. Univ. **13** (1958), no. 22, 80–92. (Russian. English summary)

Using the previously developed clothed-particle techniques [Novožilov and A. I. Terencev, Soviet Physics. JETP **36** (**9**) (1959), 89–96; Novožilov, ibid. **4** (1957), 553–561; **5** (1957), 138–140; MR **19**, 921, 923] adiabatical and velocity-dependent potentials are derived for the nucleon-hyperon interactions. Asymptotic expansions of these potentials are also discussed.

P. Roman (Manchester)

1181:

Destouches, Jean-Louis. Propriété limite de l'onde moyenne en théorie fonctionnelle des systèmes de corpuscules. C. R. Acad. Sci. Paris **248** (1959), 2722–2724.

Sous cinq conditions, l'onde moyenne d'un grand nombre de corpuscules en théorie fonctionnelle non linéaire obéit à une équation linéaire identique à l'équation d'onde de la mécanique ondulatoire usuelle.

Résumé de l'auteur

1182:

Drukarev, G. F. Approximate expressions for slow electron-atom scattering amplitude. Vestnik Leningrad. Univ. **13** (1958), no. 22, 65–74. (Russian. English summary)

The Born approximation on the one hand, and the author's previously developed integral-equation method [G. F. Drukarev, Soviet Physics. JETP **4** (1957), 309–320; MR **19**, 505] for obtaining approximative expressions for slow electron-atom scattering amplitudes are compared and the relationship between these two methods is established.

P. Roman (Manchester)

1183:

Majumdar, S. Datta. A supplement to the paper on "The problem of three bodies in quantum mechanics". Z. Physik **153** (1959), 653–654.

The paper is the revised form of the second half of § 3, pp. 535–536 of same Z. **131** (1952), 528–537 [MR **14**, 227]. After incorporating an important property of the symmetric rotor functions the theory can work with top functions as with Jackson's angular tensors.

M. Pinl (Cologne)

1184:

Mang, H. J.; und Wild, W. Zum Drei- und Vierkörperproblem der Kernphysik. Z. Physik **154** (1959), 182–217.

The Brueckner model of an atomic nucleus has been applied to nuclei consisting of three or four nucleons, using hard-core, spin- and charge-independent central two-body forces and a perturbation approach. The numerical results obtained for the binding energies of He^4 and H^3, for the energy difference between He^3 and H^3 and for the radius of He^4 are in fair agreement with experimental data.

D. ter Haar (Oxford)

RELATIVITY

See also 694, 1146, 1177, 1178, 1210, 1213.

1185:

Winogradzki, J. Invariance et conservation en théorie des champs. Application de la théorie générale à l'étude d'un champ spinoriel. Cahiers de Phys. **13** (1959), 17–26.

This paper is a general review of the application of Noether's theorem to derive conservation laws for the classical theory of fields.

P. Roman (Manchester)

1186:

Shibata, Takashi. Transformation of the electromagnetic null field under the new Lorentz subgroup, and the momentum energy of photon radiated from moving particle. Mem. Fac. Engrg. Hiroshima Univ. **1** (1958), 93–101.

The ∞^3 Lorentz transformations without rotation do not form a 3-parameter group. The author tries to modify the Lorentz transformations without rotation in order that the modified transformations form a 3-parameter subgroup G_3 of the general Lorentz group, regarding the three components of the relative velocity of one observer to the other as parameters.

J. L. Synge showed that for an electromagnetic null field which is defined by $F_{\alpha\beta}F_{\alpha\beta}=0$, $F_{\alpha\beta}F_{\alpha\beta}^*=0$, the relation between E_l, H_l and E_l', H_l' takes on a simple form (E_l electric 3-vector, H_l magnetic 3-vector, $F_{\alpha\beta}$ electromagnetic tensor, $F_{\alpha\beta}^*$ the dual tensor of $F_{\alpha\beta}$). The author considers the transformation of the electromagnetic null field under the subgroup G_3. Some interesting results: the Doppler effect for light is deduced directly by considering the transformation of the subgroup G_3. It can be proved that the momentum energy 4-vector for a photon radiated from a moving particle depends on the velocity of the particle. The exact expression for this momentum 4-vector is obtained by means of the subgroup G_3.

M. Pinl (Cologne)

1187:

Park, David. Relativistic mechanics of a particle. Amer. J. Phys. **27** (1959), 311–313.

1188:

Halbwachs, F.; Hillion, P.; and Vigier, J.-P. Lagrangian formalism in relativistic hydrodynamics of rotating fluid masses. Nuovo Cimento (10) **10** (1958), 817–833. (Italian summary)

In a paper by D. Bohm and J.-P. Vigier [Phys. Rev. (2) **109** (1958), 1882–1891; MR **20** #3018], a theory was given of the hydrodynamics of a finite mass of fluid ("droplet"), treated from the standpoint of special relativity. The paper under review investigates Lagrangian formulations of this theory.

If the droplet is supposed to be composed of non-interacting particles with 4-velocity v_μ, the density of momentum g_μ and the density of angular momentum $m_{\mu\nu}$, suitably defined, satisfy

$$\text{(I)} \qquad \dot{g}_\mu = 0, \quad \dot{m}_{\mu\nu} = g_\mu v_\nu - g_\nu v_\mu,$$

where dot means differentiation along a streamline. The set (I) is 10 equations for 13 independent unknowns. The aim is to construct a Lagrangian from which (I) will appear as conservation relations. The Euler equations of the Lagrangian will then give additional relations which can remove the indeterminacy of (I), and may give further information about the internal motions of the droplet. Two such Lagrangians are considered in detail.

W. B. Bonnor (London)

1189:

Halbwachs, Francis; et Vigier, Jean-Pierre. Lagrangien d'une masse fluide relativiste libre. C. R. Acad. Sci. Paris **248** (1959), 1124–1126.

This work is a continuation of that described in the previous review. A generalised inertia tensor $I_{\alpha\beta\ \mu\nu}$ is introduced, and from it is formed the moment tensor $M_{\mu\nu} = I_{\alpha\beta\mu\nu}\omega_{\alpha\beta}$, where $\omega_{\alpha\beta}$ is the generalised angular velocity. A Lagrangian containing $I_{\alpha\beta\mu\nu}$ is constructed, and the equations following from it are used to show that $M_{\mu\nu}$ is equal to the angular momentum tensor previously introduced into the theory in a different way.

W. B. Bonnor (London)

1190:

Kléman, Maurice. Étude de la vitesse unitaire en chaque point d'une masse fluide relativiste en rotation. C. R. Acad. Sci. Paris **248** (1959), 1126–1129.

This work concerns the hydrodynamics of a fluid droplet, as described by the theory of Bohm and Vigier [Phys. Rev. (2) **109** (1958), 1882–1891; MR **20** #3018]. The author studies the velocity v_μ of the fluid at any point by means of a frame moving with the centre of mass of the droplet. He finds an expression for v_μ which contains classical terms, and an additional term due to the deformation of the droplet during its motion (see also the two articles cited below). *W. B. Bonnor* (London)

1191:

Kléman, Maurice. Sur la notion d'axe central en hydrodynamique relativiste des masses fluides en rotation. C. R. Acad. Sci. Paris **247** (1958), 1718–1720.

1192:

Kléman, Maurice. Sur la notion de spin en hydrodynamique relativiste des masses fluides en rotation. C. R. Acad. Sci. Paris **247** (1958), 1831–1833.

1193:

Infeld, L. Equations of motion and gravitational radiation. Ann. Physics **6** (1959), 341–367.

The non-existence of gravitational radiation is established in the following (rather specialised) sense. Let $g^{\mu\nu} = \eta^{\mu\nu} + \gamma^{\mu\nu}$ ($\gamma^{\mu\nu}$ not necessarily small), and write the Einstein tensor-density

$$\mathfrak{G}^{\mu\nu} = S^{\mu\nu\alpha\beta}{}_{,\alpha\beta} + \Lambda^{\mu\nu},$$

where

$$S^{\mu\nu\alpha\beta} = \tfrac{1}{2}[\gamma^{\mu\beta}\eta^{\nu\alpha} + \gamma^{\nu\alpha}\eta^{\mu\beta} - \gamma^{\alpha\beta}\eta^{\mu\nu} - \gamma^{\mu\nu}\eta^{\alpha\beta}],$$

and the $\Lambda^{\mu\nu}$ includes all non-linear expressions in the $\gamma^{\mu\nu}$. Treating the sources of the field as singularities with energy tensor-density of the form $\mathfrak{T}^{\mu\nu} = \sum m\dot{\xi}^\mu\dot{\xi}^\nu\delta$, the identity (in virtue of the field equations)

$$\frac{1}{8\pi}\int \Lambda^{\alpha a} d^3x + m\dot{\xi}^\alpha = \frac{1}{8\pi}\int S^{m\alpha a\beta}{}_{,\beta} n_m dS$$

(Latin suffixes 1, 2, 3) is used to identify the terms occurring in it as, respectively, field, inertial and gravitational linear momenta. The last-named is invariant under a small change in the coordinate-system; and the flux of radiation is defined as its time rate-of-change.

It is then shown that if we can impose at infinity the coordinate conditions $\gamma^{0\alpha}{}_{,\alpha} = O(r^{-3})$, $\gamma^{mn}{}_{,n} = O(r^{-3})$, we can always find a reasonable coordinate-system in which there is no gravitational radiation.

C. W. Kilmister (London)

1194:

Bel, Louis. Introduction d'un tenseur du quatrième ordre. C. R. Acad. Sci. Paris **248** (1959), 1297–1300.

The four-suffix "super-energy" tensor of the gravitational field defined in an earlier note [same C. R. **247** (1958), 1094–1096; MR **20** #6307] is generalised to the case when $R_{\alpha\beta} \neq \lambda g_{\alpha\beta}$. If

$$R_{\gamma\delta,}{}^{\alpha}{}_{\sigma;\alpha} = J_{\gamma\delta,\sigma} \equiv R_{\delta\sigma;\gamma} - R_{\gamma\sigma;\delta},$$

and if

$$S_{\alpha\beta\lambda\mu} = \tfrac{1}{4}\eta_{\alpha\beta\rho\sigma}\eta_{\lambda\mu\gamma\delta}R^{\rho\sigma\gamma\delta}$$

is the double dual of $R_{\alpha\beta\lambda\mu}$, then define

$$T^{\alpha\beta,\lambda\mu} = R^{\alpha\rho,\lambda\sigma}R^{\beta}{}_{\rho,}{}^{\mu}{}_{\sigma} + R^{\alpha\rho,\mu\sigma}R^{\beta}{}_{\rho,}{}^{\lambda}{}_{\sigma} + S^{\alpha\rho,\lambda\sigma}S^{\beta}{}_{\rho,}{}^{\mu}{}_{\sigma} + S^{\alpha\rho,\mu\sigma}S^{\beta}{}_{\rho,}{}^{\lambda}{}_{\sigma} - \tfrac{1}{4}g^{\alpha\beta}g^{\lambda\mu}R_{\gamma\delta,\rho\sigma}R^{\gamma\delta,\rho\sigma}.$$

This tensor is completely symmetric, and if u_α is a time-like unit vector, $T^{\alpha\beta,\lambda\mu}u_\alpha u_\beta u_\lambda u_\mu$ is positive or zero, as in the previous note. Now, however,

$$T^{\alpha\beta,\lambda\mu}{}_{;\alpha} = R^{\beta}{}_{\rho,}{}^{\mu}{}_{\sigma}J^{\lambda\sigma,\rho} + R^{\beta}{}_{\rho,}{}^{\lambda}{}_{\sigma}J^{\mu\sigma,\rho} + \tfrac{1}{2}[S_{\alpha\rho}{}^{\lambda}{}_{\sigma}G^{\alpha\beta\rho,\mu\sigma} + S_{\alpha\rho}{}^{\mu}{}_{\sigma}G^{\alpha\beta\rho,\lambda\sigma}],$$

where

$$G_{\alpha\beta\rho,}{}^{\mu}{}_{\sigma} \equiv S_{\beta\rho}{}^{\mu}{}_{\sigma;\alpha} + S_{\rho\alpha}{}^{\mu}{}_{\sigma;\beta} + S_{\alpha\beta}{}^{\mu}{}_{\sigma;\rho}.$$

C. W. Kilmister (London)

1195:

Anderson, James L. Superpotentials for a generally covariant field theory. Phys. Rev. (2) **112** (1958), 1826–1827.

The author's summary reads as follows. "An expression for the superpotentials corresponding to the energy-momentum complex of a generally covariant field theory is presented. It is somewhat more general than previous expressions in that it makes no special assumptions regarding the symmetry properties of the Lagrangian of the theory." *W. B. Bonnor* (London)

1196:

Komar, Arthur. Covariant conservation laws in general relativity. Phys. Rev. (2) **113** (1959), 934–936.

Hitherto attempts to formulate a meaningful and unique expression for energy density in general relativity theory have always proved to be inconclusive. Recently, however, Møller [C. Møller, Ann. Physics **4** (1958), 347–371; MR **20** #732] has shown how to remedy the difficulty encountered by the lack of covariance of the Einstein pseudotensor, by altering the computed values for the energy density and the total energy. Also Bergmann [P. G. Bergmann, Phys. Rev. (2) **112** (1958), 287–289; MR **20** #5677] has indicated that the Einstein pseudotensor may be thought of as the set of generators of the various infinitesimal coordinate transformations corresponding to the rigid parallel displacements of the coordinate origin. He suggests that we may expect a significant conserved quantity to correspond to each infinitesimal coordinate transformation.

In the present paper Komar begins by reviewing the work of Møller and Bergmann and then proceeds to combine their results. A set of covariant conservation laws is constructed and their relationship to the generators of infinitesimal coordinate transformations is indicated. It is shown that, in all coordinate systems, certain of these quantities may be naturally identified as energy and momentum. *G. L. Clark* (London)

1197:

Riabushko, A. P.; and Fisher, I. Z. The motions of rotating masses in the general theory of relativity. Soviet Physics. JETP **34** (7) (1958), 822–825 (1189–1194 Ž. Eksper. Teoret. Fiz.).

The authors study the two-body problem for rotating masses, using the approximate equations of motion derived from Einstein's field equations by Riabushko in the paper reviewed below. Their results are as follows.

(i) The rotation of a body about its own axis. In the first non-Newtonian approximation, the spin-moment is constant in magnitude but precesses about the normal to the plane of the Newtonian orbit with a period

$$\left[\frac{G}{C^3}\frac{m_2(2m_1+m_2)}{(m_1+m_2)p}\right]^{-1}\tau,$$

where the suffix 1 refers to the body concerned, p is related to the Newtonian orbital motion by $\{G(m_1+m_2)p\}^{\frac{1}{2}}=\gamma^2\dot{\varphi}$, and τ is the orbital period. In the second non-Newtonian approximation the magnitude of the spin-moment changes secularly.

(ii) The orbital angular momentum. This undergoes a secular deviation from its Newtonian value. In the first approximation its magnitude does not change, but there is a secular deviation of the plane of the relativistic orbit from the Newtonian plane (unless both spin moments are normal to the orbital plane).

(iii) Rotation of the perihelion. The general result is complicated and the authors restrict themselves to the case when the spin moments are normal to the orbital plane, which is then fixed. The spin moments produce a rotation of the perihelion, the formula for which is too long to quote here. Its ratio to the familiar rotation of perihelion is of the same order as the ratio of the spin moment to the orbital moment (which in practical cases is usually negligible).

An additional result is that the Newtonian energy does not undergo a secular change. The authors state that this remains true for arbitrary orientations of the spin moments.

(iv) The Newtonian centre of inertia. This undergoes a secular acceleration. However, the authors believe that this result may be a spurious consequence of their approximation methods. "The question as to what is the true motion of the Newtonian centre of inertia over long periods of time remains open." *D. W. Sciama* (London)

1198:

Riabushko, A. P. Equations of motion for rotating masses in the general theory of relativity. Soviet Physics. JETP **33** (6) (1958), 1067–1073 (1387–1395 Ž. Eksper. Teoret. Fiz.).

The author uses the Infeld δ-function technique [L. Infeld, Acta Physica Polonica **13** (1954), 187–204; MR **16**, 531] to derive approximate (sixth-order) equations for the translational and rotational motion of rotating masses. Both these motions contain spin-orbit and spin-spin interactions. The approximate equations of motion have been integrated by the author and I. Z. Fisher in the paper reviewed above. *D. W. Sciama* (London)

1199:

Pham, Mau Quan. Induction électromagnétique en relativité générale. Cahiers de Phys. **12** (1958), 297–307.

L'A. définit, dans un milieu fluide de vitesse unitaire u^α, permittivité ε et perméabilité μ, la métrique associée à la métrique d'espace temps $ds^2=g_{\alpha\beta}dx^\alpha dx^\beta$ par:

$$\overline{ds}^2=[g_{\alpha\beta}-(1-(\varepsilon\mu)^{-1})u_\alpha u_\beta]dx^\alpha dx^\beta;$$

il montre que les géodésiques de longeur nulle de cette métrique sont les rayons électromagnétiques et que les équations de Maxwell peuvent s'ecrire:

$$\nabla_\alpha\dot{H}^{\alpha\beta}=0,\quad \nabla_\alpha\bar{G}^{\alpha\beta}=J^\beta,\quad \bar{G}^{\alpha\beta}=\mu^{-1}\bar{H}^{\alpha\beta}$$

($H_{\alpha\beta}$ et $G_{\alpha\beta}$ sont les tenseurs champ électrique-induction magnétique et champ magnétique-induction électrique). L'intégration par parties de ces équations le conduit à prendre pour tenseur de Maxwell le tenseur symétrique

$$t^{\alpha\beta}=\bar{\tau}^{\alpha\beta}=\tau^{\alpha\beta}-(1-\varepsilon\mu)\tau^{\alpha\rho}u_\rho u_\beta$$

($\tau^{\alpha\beta}$ tenseur de Minkowski). L'A. dit qu'une induction est singulière au sens de Lichnerowicz si:

$$\tfrac{1}{2}G_{\alpha\beta}H^{\alpha\beta}=0,\quad \tfrac{1}{2}G_{\alpha\beta}\dot{H}^{\alpha\beta}=0,$$

et montre qu'alors $t^{\alpha\beta}=\xi^2 l_\alpha l_\beta$, les trajectoires du vecteur l_α étant les rayons électromagnétiques on a ainsi une représentation des photons.

L'A. montre d'autre part que tout schéma induction singulière peut s'interpréter comme un schéma matière pure et vice-versa. *Y. Fourès-Bruhat* (Marseille)

1200:

Rayner, Charles Beresford. Mouvement rigide en relativité générale. C. R. Acad. Sci. Paris **248** (1959), 929–932.

L'A. montre que si le vecteur vitesse d'un fluide en mouvement satisfait aux équations du mouvement rigide de Rosen la densité propre et le carré du moment angulaire restent constants le long des lignes de courant.
Y. Fourès-Bruhat (Marseille)

1201:

Cattaneo, Carlo. Dérivation transverse et grandeurs relatives en relativité générale. C. R. Acad. Sci. Paris **248** (1959), 197–199.

L'A. introduit des grandeurs relatives, invariantes par les changements de coordonnées conservant les lignes d'univers temporelles où x^4 varie seul. Il écrit l'équation du mouvement d'une particule d'épreuve au moyen de ces grandeurs sous forme newtonienne, faisant apparaître au deuxième membre la somme de trois champs (à potentiel scalaire, à potentiel vecteur, et un analogue du champ de Coriolis). *Y. Fourès-Bruhat* (Marseille)

1202:

Souriau, Jean-Marie. Conséquences physiques d'une théorie unitaire. C. R. Acad. Sci. Paris **248** (1959), 1478–1480.

The five-dimensional theory proposed earlier [same C. R. **247** (1958), 1559–1562; MR **20** #6312] is considered in more detail in the special relativity approximation. Certain characteristics of the group under which the theory is invariant are related to parity, charge and mass.

C. W. Kilmister (London)

ASTRONOMY

See also 995, 996, 997, 1121, 1122.

1203:

Alfven, H. The origin of the solar system. Voprosy Kosmog. **6** (1958), 78–97. (Russian. English summary)

A short review of the author's papers on the origin of the solar system—a theory which takes into account the important role of the electromagnetic forces.

Author's summary

1204:

Idlis, G. M. The symmetry of stationary axial-symmetrical stellar systems relative to the equatorial plane. Astr. Ž. **36** (1959), 85–88. (Russian. English summary)

The aim of this note has been to prove (by a method essentially analogous to Lichtenstein's) that the stationary symmetrical self-gravitating systems consisting of an aggregate of points of finite mass are bound to exhibit symmetry with respect to the equatorial plane of rotation of the system. *Z. Kopal* (Manchester)

1205:

Vlasov, A. A. On spatially inhomogeneous distributions of a system of gravitating particles. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 4, 95–107. (Russian)

The author departs from the nonlinear equations governing the equilibrium structure of a stationary system of mass points (section 1), and considers (in section 2) their solution for certain particular types of symmetry (plane, or cylindrical) of the system, leading to ordinary differential equations. Section 3 is devoted to a discussion of particular solutions characterized by spiral structure; and the concluding section 4 contains some remarks on the phenomena arising if the presence of charged particles produces an electrostatic field.

Z. Kopal (Manchester)

1206:

Grémillard, Jean. Sur un cas particulier de solutions périodiques de troisième sorte du problème des trois corps. C. R. Acad. Sci. Paris **247** (1958), 1307–1310.

With reference to two previous notes [same C. R. **239** (1954), 153–155; **244** (1957), 1011–1014; MR **16**, 181; **18**, 857] the author proves in the present note the existence of periodic solutions of the third kind in the case of small inclinations of the osculating orbits about the central body. The motions in these orbits may have the same sense or opposite senses. Such solutions exist no matter what may be the initial values of the mean longitudes of the two planets. *E. Leimanis* (Vancouver, B.C.)

1207:

Ščigolev, B. M. Approximate calculation of ephemerides in an unperturbed elliptic motion. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 4, 45–56. (Russian)

The author introduces simply harmonic expressions for the rectangular coordinates of the relative position of a body revolving in an elliptic orbit, whose coefficients are suitable time-averages of the corresponding exact terms; and shows that the errors remain reasonably small for bodies moving in mildly eccentric orbits.

Z. Kopal (Manchester)

1208:

Agekian, T. A. Interaction of stars with diffuse matter. Voprosy Kosmog. **6** (1958), 221–237. (Russian. English summary)

An equation expressing the acceleration of a star due to its interaction with a system of fixed dust clouds is obtained. Gravitation and light pressure are taken into consideration. Accelerations of low luminosity stars are very low and negative. Accelerations of high luminosity stars are positive and may be large. An equation expressing the acceleration of stars during their passage through a gas cloud is deduced for hot giants, when the gravitation of gas caused by the star is negligible as compared with the light pressure. In this case the acceleration is also positive and essential. The acceleration of hot giants caused by their interaction with diffuse matter can explain certain well-known phenomenon.

From the author's summary

1209:

Sobolev, V. V. Physics of planetary nebulae. Voprosy Kosmog. **6** (1958), 112–156. (Russian. English summary)

A brief survey of the modern state of physics of planetary nebulae is given. The following questions are considered: (1) temperatures of nuclei, (2) theory of Balmer decrement, (3) forbidden lines in the spectra of nebulae, (4) electron temperatures, (5) electron densities, (6) chemical composition of nebulae, (7) continuous spectrum of nebulae, (8) diffusion of L_α-radiation, (9) radiation pressure, (10) structure of planetary nebulae, (11) problem of the origin of planetary nebulae. Some theoretical works are criticized. Problems for future investigations are proposed.

Author's summary

1210:

★**Баум, Ф. А.; Каплан, С. А.; и Станюкович, К. П. Введение в космическую газодинамику. [Baum, F. A.; Kaplan, S. A.; and Stanyukovič, K. P. Introduction to cosmic gas dynamics.]** Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 424 pp. 13.50 rubles.

The volume under review constitutes an excellent introduction to, and a survey of the most important results of, the study of its subject as indicated by the title—without parallel in any Western language. Its contents is divided in three essential parts. The first, entitled "Cosmic Gas Dynamics", divided in 9 chapters, contains a quite thorough survey of the equations of fluid flow and of their solution in a diversity of problems—with special reference to the occurrence and propagation of shock waves, of their structure as influenced by temperature (dissociation, ionization, etc.). Much of this first part deals with the fundamental properties of supersonic gas flow; self-gravitating (cosmic) gas does not make explicit appearance until at a relatively late stage in Chapter 7. The penultimate Chapter 8 then deals with an application of gas dynamics to certain problems of non-stable stars; while the last Chapter 9 is devoted to problems occurring in geophysics (and, in particular, to the gas compression phenomena accompanying the flight and impact of meteorites).

The second part, in five chapters, gives an outline of the mathematical elements of magneto-hydrodynamics. The fundamental equations are described, and their solutions (in particular, the similarity solutions) indicated in simple cases. Close attention is paid to the behaviour of shock waves in a magnetic field (in media of infinite as well as finite electrical conductivity). Chapters 4 and 5 are devoted to an outline of a statistical theory of isotropic hydromagnetic turbulence and to a study of its spectrum.

The last part (in three chapters) deals with the fundamentals of relativistic magneto-hydrodynamics, including the relativisitic effects on shock waves propagating at high speed. The recondite character of its subject did not affect the clarity of presentation, which remains on a high level throughout the whole volume, and bespeaks the amount of expository forethought. Each one of the three parts is, moreover, concluded by a bibliography, which is sparing in references to non-Russian sources (particularly in part I), but reflects the wealth of original papers currently published in this field in the Soviet Union and many of which are but little known to non-Russian readers. *Z. Kopal* (Manchester)

1211:

Kaplan, S. A. Magnetogasdynamics and problems of cosmogony. Voprosy Kosmog. **6** (1958), 238–264. (Russian. English summary)

In this paper the following basic principles and main results, obtained in magnetogasdynamics, are given: the existence of the "freezing-in" integral, the description of "entanglement" and "unentanglement" of the magnetic lines of force, the increase of magnetic energy in magnetogas shock waves and the concept of magnetogas turbulence. Some cosmogonic hypotheses based on methods and results of magnetogasdynamics are discussed: (a) the connection of spiral arms with the regular magnetic field; (b) the formation of interstellar gas clouds as separate whirl-units of interstellar magnetogas turbulence; (c) the formation of the "halo" of the galaxy as a result of magnetic pressure and that of cosmic rays on the magnetized interstellar gas; (d) the formation of filamentary nebulae as a result of the possible instability of magnetogas shock waves; (e) the origin of stellar magnetic fields in the process of condensation; (f) the acceleration of cosmic ray particles in the interstellar magnetogaseous medium. *Author's summary*

1212:

Ogorodnikov, K. F. Statistical mechanics of galaxies with non-homogeneous stellar population. Astr. Ž. **35** (1958), 408–423. (Russian. English summary)

The method of additive parameters is applied to stellar systems of Kapteyn-Lindblad type, that is systems composed of subsystems with different stellar populations and from a dynamical point of view characterized by different rotational speeds and different dispersions of the residual velocities. In the case of a homogeneous stellar population, treated by the author in a previous paper [same Ž. **34** (1957), 770–789; MR **19**, 1239], three additive parameters occur, the mass, the sum of the energies of motion of the stars and the sum of the angular momentums of the stars. In the case of Kapteyn-Lindblad systems a fourth parameter is added, namely the sum of squares of the angular momentums of the stars with respect to the rotational axis of the galaxy. Moreover, the mean speed Θ_0 in the direction of the rotational motion is supposed to be a function of the distance ρ from the axis of rotation according to the empirical law $\Theta_0 = k_1\rho/(1+k_2\beta^2)$, where k_1 and k_2 are constants.

The following formula for the first approximation ν_0 of the stellar density is derived:

$$\nu_0 = \frac{\kappa_3^2}{2\pi G m}\,\frac{(1-\kappa_4\rho^2)}{(1+\kappa_4\rho^2)^3},$$

where κ_3 and κ_4 are constants. The value of the distance ρ for which ν_0 vanishes is interpreted as the boundary of the nucleus of the galaxy. A second approximation to the stellar density is derived with the aid of methods similar to those of the previous paper referred to above.

Lastly it is stated in a qualitative way how spiral galaxies can be developed from elongated elliptical galaxies. The stars dissipate, but the diffuse matter is retained by the magnetic field and a secondary process of star formation sets in. Finally the spiral arms disappear and an E-galaxy remains. *E. Lyttkens* (Uppsala)

1213:

Nariai, Hidekazu; and Ueno, Yoshio. On a new approach to cosmology. Progr. Theoret. Phys. **21** (1959), 219–231.

Since the Robertson-Walker metrics involve only one function, $\phi(t)$, instead of ten functions g_{ij}, the authors propose the one field-equation (1) $R = kT$ for cosmological theory. To allow for local condensation processes (galaxy formation), they generalise the R-W metrics by considering metrics conformal to a space ds_0^2 of constant curvature: $ds^2 = \psi^2(t, r, \theta, \varphi)ds_0^2$. The function ψ (which satisfies (1)) is equivalent to $\phi(t)$ when it is a function of t only; in the general case it can be used to describe local gravitational phenomena.

The authors proceed to solve their field equations when $\psi = \psi(t)$. They obtain some model universes inconsistent with Einstein's equations, e.g. a de Sitter metric with non-vanishing density of matter (steady state model).

This is possible because the field equations (1) do not imply conservation equations for T_{ij}.

The authors then reject those of their models inconsistent with Baum's observational result [Astr. J. **62** (1957), 6–7] that the acceleration parameter $\ddot{\phi}\phi/\dot{\phi}^2$ is negative.

{This theory is similar to one proposed by D. E. Littlewood [Proc. Cambridge Philos. Soc. **49** (1953), 90–96; MR **14**, 592] which Pirani showed [ibid. **51** (1955), 535–537; MR **16**, 1165] leads to the wrong value for the advance of perihelion. In addition, Baum's result is unreliable because of selection effects [Scott, Astr. J. **62** (1957), 248–265].} *D. W. Sciama* (London)

1214:

Bass, R. W.; and Witten, L. Remark on cosmological models. Rev. Mod. Phys. **29** (1957), 452–453.

The following theorem is proved. Let V_n be an n-dimensional differentiable manifold, and let F be a continuous, class C^1 vector field defined on V_n. Suppose that F vanishes at most once and that its curl vanishes identically on V_n. Then, either V_n is noncompact, or V_n is compact and its first Betti number does not vanish.

It is suggested that such vector fields might be used to study the topology of cosmological models.

W. B. Bonnor (London)

1215:

Naan, G. Über den gegenwärtigen Stand der kosmologischen Wissenschaft. Voprosy Kosmog. **6** (1958), 277–329. (Russian. German summary)

Die Kosmologie wird als ein der Astronomie, der theoretischen Physik und der Philosophie angrenzendes Wissensgebiet betrachtet. *Zusammenfassung des Autors*

1216:

Haser, L. Distribution d'intensité dans la tête d'une comète. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 740–750.

The radial distribution of molecules in the head of a comet is investigated theoretically. Formulae for comparison with the observations are given.

Author's summary

1217:

Rohrlich, F. Theoretical multiplet strengths. Astrophys. J. **129** (1959), 441–448.

General formulae are derived for multiplet strengths of various transitions of astrophysical interest. The formulae are derived by a straightforward application of the methods of tensor algebra introduced by Racah and are expressed in terms of fractional percentage and Racah coefficients. The results provide a considerable extension of previous tables of multiplet strengths.

A. Dalgarno (Cambridge, Mass.)

1218:

Rohrlich, F. Sum rules for multiplet strengths. Astrophys. J. **129** (1959), 449–455.

Various sum rules for multiplet strengths between electron configurations are derived by the methods of tensor algebra introduced by Racah. Some errors in the earlier literature are corrected.

A. Dalgarno (Cambridge, Mass.)

GEOPHYSICS

See also 1073, 1074, 1096.

1219:

Belousov, S. L. Estimation of errors of approximation in solving the equation of vortex transport by the method of successive approximations. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1958**, 1145–1150. (Russian)

The estimate concerns the meteorological problem

$$\partial\nabla^2 z/\partial t = -(g/l)\partial(z, \nabla^2 z)/\partial(x, y) - \beta\partial z/\partial x;$$

$$z = -Cy + D\cos(\mu x + \nu y) \quad (t = 0, -\infty < x, y < \infty),$$

that is, the change in time of the initial shape $z_{t=0}$ of an isobaric surface in the atmosphere.

G. Kuerti (Cleveland, Ohio)

1220:

Andreoletti, Jean. Dynamique et statistique en météorologie. J. Sci. Météorol. **10** (1958), 77–86. (English and Spanish summaries)

1221:

Gillert, H. The bearing problem in space with range variation. Vestnik Leningrad. Univ. **14** (1959), no. 7, 143–145. (Russian. English summary)

A simple confidence ellipsoid for the 3-dimensional problem of taking bearings with normal unbiased errors is constructed. *Author's summary*

1222:

Gillert, H. Estimation of horizontal angles by the Gauss-Schreiber method. (Confidence ellipsoid). Vestnik Leningrad. Univ. **14** (1959), no. 7, 140–142. (Russian. English summary)

The confidence ellipsoid for the simultaneous estimation of horizontal angles by the Gauss-Schreiber method is constructed. *Author's summary*

OPERATIONS RESEARCH, ECONOMETRICS, GAMES

See also 808.

1223:

★Saaty, Thomas L. Mathematical methods of operations research. McGraw-Hill Book Co., Inc., New York–Toronto–London, 1959. xi+421 pp. $10.00.

An introductory text in operations reseach, which includes a section on methodology, as well as short discussions of some of the mathematical models useful in describing operations. After the section on methodology there is one on optimization procedures, such as linear and quadratic programming and game theory. Another section is on stochastic models, including queuing theory. Examples are given of the application of these theories to operational situations. A sizeable number of problems is listed. *P. M. Morse* (Cambridge, Mass.)

1224:

Groves, Arthur D.; and Smith, Ed S. Salvo hit probabilities for offset circular targets. Operations Res. **5** (1957), 222–228.

Si suppone circolare normale la distribuzione dei punti

di collisione d'una salva di n missili, con centro $C.I._B$ e scarto standard σ_B; di ugual tipo la distribuzione dei centri $C.I._B$, con centro $C.I._A$ e scarto standard σ_A; infine il bersaglio si suppone circolare, con centro e raggio dati. La determinazione della probabilità di colpire il bersaglio, invece di essere ottenuta per mezzo di un integrale doppio (della distribuzione dei missili sopra l'area del bersaglio) è ricondotta ad una integrazione semplice rispetto alla distanza dal centro del bersaglio, riferendo la distribuzione a questo invece che al suo proprio centro. Il metodo seguito, come dichiarano gli AA., è più numerico che analitico; sono fornite utili tabelle per i più comuni valori di σ_A, σ_B, grafici di uso pratico (equivalenti alle tabelle), e semplici esempi. *D. Fürst* (Rome)

1225:

Beale, E. M. L. An algorithm for solving the transportation problem when the shipping cost over each route is convex. Naval Res. Logist. Quart. **6** (1959), 43–56.

The paper treats a variant of the capacitated Hitchcock problem, in which the flows along the routes linking sources to destinations are required to be non-negative integers and the shipping cost for a route is a convex function of the amount shipped over this route. On account of this convexity of the cost function, a shipping program that cannot be improved by changing the flow along each route by at most one unit cannot be improved at all. An algorithm is presented for solving this problem. While this algorithm could also be used when the cost function is linear, it does not seem to be more advantageous than the standard methods in this case.

W. Prager (Providence, R.I.)

1226:

Gerstenhaber, Murray. A solution method for the transportation problem. J. Soc. Indust. Appl. Math. **6** (1958), 321–334.

The Hitchcock-Koopmans transportation problem can be combined with its dual program and stated as follows: Given an m by n matrix (c_{ij}) and two positive vectors (a_i) and (b_j) such that $\sum_i a_i = \sum_j b_j$, find a matrix (x_{ij}) and a vector (λ_i) such that (1) $x_{ij} \geqq 0$, (2) $\sum_j x_{ij} = a_i$, (3) $\sum_i x_{ij} = b_j$, (4) $x_{ij} > 0$ implies $c_{ij} - \lambda_i = \min_k (c_{kj} - \lambda_k)$ hold for all free values of the indices i and j.

In the present paper, the λ_i are interpreted as "subsidies" which reduce the unit transportation costs c_{ij} of producer i when supplying consumer j. The problem posed above becomes the following: Find non-negative quantities (x_{ij}) which exhaust the producers' supplies (a_i), satisfy the consumers' demands (b_j), and are such that each consumer j is supplied at the minimum subsidized transportation cost $\min_k (c_{kj} - \lambda_k)$. It is an obvious consequence of the duality theorem of linear programming that this problem has a solution and that the resulting quantities (x_{ij}) minimize the total transportation cost $\sum_{i,j} c_{ij} x_{ij}$ subject to (1), (2), and (3). The resulting λ_i are termed equilibrating subsidies.

The proposed solution method is cast in terms of consumer ordering policies $x_{ij} = x_{ij}(\lambda)$ which depend on the subsidies λ_i and a non-negative "threshold" r. These policies are to satisfy (1) and (3), while (4) is replaced by (4′): $x_{ij} > 0$ implies $c_{ij} - \lambda_i \leqq \min_k (c_{kj} - \lambda_k) + r$. If λ_i are subsidies such that $x_i(\lambda) = \sum_j x_{ij}(\lambda) = a_i$, $i = 1, \cdots, m$, then $x_{ij}(\lambda)$ is a feasible solution to the transportation problem and, if r is the value of the threshold, then $r \sum_i a_i$ is an upper bound for the difference between $\sum_{i,j} c_{ij} x_{ij}(\lambda)$ and the minimum feasible transportation cost. The basic computational problem attacked in the paper is the solution of the system $x_i(\lambda) = a_i$ for small values of r. Machine experience is compared with other methods.

H. W. Kuhn (London)

1227:

Wagner, Harvey M. On a class of capacitated transportation problems. Management Sci. **5** (1959), 304–318.

The author's summary: "Transportation models (ordinary and transhipment) having certain types of capacity constraints on the flows between origins and destinations are studied from the point of view of transforming them into enlarged standardized (non-capacitated) models. Specifically constraints on the flow from disjoint and/or nested sets of origins to all destinations, and from any single origin to disjoint and/or nested sets of destinations are considered. Dual formulations are indicated for constraints on the flow to destinations from origins. In the case of a set of capacity constraints on the flow from each origin to each destination, the models proposed are easily seen to be of minimal dimension for any 'standardized' version of such a capacitated problem."

The major difficulty with the suggested method is the tremendous increase in size of the equivalent uncapacitated problem. Some special cases of the above formulation can in fact be handled without enlarging the size by observation of cost dominances between competing individual routes and forming rows corresponding to aggregated origins. *A. Charnes* (Evanston, Ill.)

1228:

Bellman, Richard. On a routing problem. Quart. Appl. Math. **16** (1958), 87–90.

A functional equation approach to the problem of determining the "shortest" path in a connected network between two given nodes. The algorithm obtained was individually obtained (as the author notes in proof) by Max Woodbury and George Dantzig without invoking Bellman's approach. *A. Charnes* (Evanston, Ill.)

1229:

Daubin, Scott C. The allocation of development funds: an analytic approach. Naval Res. Logist. Quart. **5** (1958), 263–276.

"The administrative problem of fund allocation among the several projects constituting the development program is considered from an analytic-economic viewpoint. The concept of operational potential is introduced and used in establishing a measure of effectiveness for the development program, the 'development contribution function'. The problem then resolves to the maximization of a mathematical expression. An example is worked out to demonstrate the method and the utility of the results." (From the author's summary)

A. Charnes (Evanston, Ill.)

1230:

Gomory, Ralph E. Outline of an algorithm for integer solutions to linear programs. Bull. Amer. Math. Soc. **64** (1958), 275–278.

This is a path-breaking contribution which multiplies tenfold the power of linear programming methods. It provides a finite, real field algorithm for solving the

discrete problem: Maximize a linear integral form among integer lattice points of a polyhedral convex set. The efficiency of the method where M linear inequalities characterize the polyhedral set is that of linear programming, i.e., the effort involved is that of inverting an $M \times M$ matrix. This is the first method which efficiently solves a reasonably general problem of finding a global maximum where many local maxima are possible. It provides a calculus which straight-forwardly handles problems of combinatorial, diophantine and Boolean analysis constructively, removing them from their heretofore character of intractable curiosities and mathematical recreations. Continuum problems involving disconnected unions of polyhedral convex sets may also be reduced to routine character.

The finiteness of the algorithm, which is the most critical element of novelty, depends on the property of finite increase at each stage, based on regularization (ε-perturbation) techniques first achieved by Charnes, application of the dual (not dual simplex) method of Lemke, and the use of Beale's form of the simplex tableau to obtain the repercussions of integrality. The references cited are inadequate to convey a proper idea of the ubiquitous possibilities for application. The possibility of formulating many important applicational problems as diophantine programming problems has been recognized for years by many other workers in the field of linear programming. However, such a formulation has hitherto been almost exclusively a dead-end.

A. Charnes (Evanston, Ill.)

1231:

Kataoka, Shinji. A method of discrete programming problem. J. Operations Res. Soc. Japan **2** (1958), 27–42.

The problem of maximising a linear function: $Z = \sum_j v_j x_j$ is considered under the following restrictions:

(a) $$\sum_j a_{ij} x_j = s_i \quad (x_j \geqq 0);$$

(b) $$\sum_i \pi_i \sum_j a_{ij} = \sum_j q_j x_j = \sum_i \pi_i s_i = R$$

(π_i constant, $x_j \geqq 0$);

(c) $0 \leqq x_j \leqq 1$; (d) $x_j = 0$ or 1. Let A be the set of solutions which satisfies (a), etc. Then: $AD \subset BD \subset BC$. A heuristic method is indicated for solving the discrete programming problem, which consists in choosing appropriate constants π_i. A criterion is indicated which shows if the optimal solution has been reached. The method is illustrated with the help of an example. *G. Tintner* (Ames, Iowa)

1232:

Parisot, Georges. Les programmes logarithmiques. Leur application au calcul des programmes linéaires. C. R. Acad. Sci. Paris **248** (1959), 1470–1472.

This paper presents a generalization of the Frisch logarithmic potential method which the author believes will permit the approximation calculation of arbitrary linear programming models involving up to about a thousand constraints. Similar replacements of linear programming problems by unconstrained maximization problems have been studied by Frisch's group, e.g., L. Johannsen, and originally (1951) by T. Motzkin and associates at the National Bureau of Standards. The author hopes his more complicated method will converge more rapidly than the Frisch scheme. It is not clear what the order of convergence will be nor is it clear how one adjusts in the course of calculation an essential vector in order that its components will not be too large and yet not be too small. As a matter of fact, the inability to be able to specify a good rule for doing this led to abandonment of early attempts by Motzkin and others. Some actual numerical studies by the author might therefore prove illuminating. *A. Charnes* (Evanston, Ill.)

1233:

Elmaghraby, Salah E. An approach to linear programming under uncertainty. Operations Res. **7** (1959), 208–216.

Let the demand for a product be a chance variable, all relevant marginal costs be constant. The demands for different products are independent. The demand distribution function is discrete and stationary. The maximization of expected profit (or minimization of expected cost) is achieved by considering increments in the amounts produced. Using these increments and the partial differences of the objective function the problem is converted into a (larger) linear programming problem of the orthodox type. There is a similarity with convex programming [A. Charnes and E. Lemke, Naval Res. Logist. Quart. **1** (1955), 301–312; MR **17**, 537]. *G. Tintner* (Ames, Iowa)

1234:

Klahr, Carl N. Multiple objectives in mathematical programming. Operations Res. **6** (1958), 849–855.

The author observes that some problems involving multiple objectives upon closer scrutiny can be reformulated by partitioning the domain of the variables in such a manner that a single (different) objective may be relevant over each element of the partition. He remarks this is often possible at least in adequate approximation.

A. Charnes (Evanston, Ill.)

1235:

Wagner, Harvey M.; and Whitin, Thomson M. Dynamic version of the economic lot size model. Management Sci. **5** (1958), 89–96.

The authors state: "A forward algorithm for a solution to the following dynamic version of the economic lot size model is given: allowing the possibility of demands for a single item, inventory holding charges, and setup costs to vary over N periods, we desire a minimum total cost inventory management scheme which satisfies known demand in every period. Disjoint planning horizons are shown to be possible which eliminate the necessity of having data for the full N periods."

If the setup cost does not vary with time, the results can be obtained as special instances of the horizon theorem of Charnes, Cooper and Mellon [Econometrica **23** (1955), 307–323; MR **17**, 58]. *A. Charnes* (Evanston, Ill.)

1236: [Schild, Alfred]

Schild, A. On inventory, production and employment scheduling. Management Sci. **5** (1959), 157–168.

"The paper extends results of Holt, Modigliani, Muth and Simon in scheduling overtime to minimize costs. It studies generalizations of the cost function which are possible without spoiling the essential simplicity of the decision rules. This is illustrated by a costing model which

includes: (a) inventories of individual products, (b) a more realistic treatment of set-up costs, (c) an interest factor to discount future costs, and (d) the cost of additional temporary storage when inventory is high." (Author's summary) *A. Charnes* (Evanston, Ill.)

1237:

Braithwaite, R. B. A terminating iterative algorithm for solving certain games and related sets of linear equations. Naval Res. Logist. Quart. **6** (1959), 63–74.

"This paper will explain, and justify the use of, an iterative algorithm, which, though applicable only to certain special classes of games, will give their exact values and optimal strategies after a predeterminate number of steps. Described in terms of fictitious plays, the iteration is effected by the players playing alternately, the player playing second always choosing a pure strategy best against his opponent's accumulated mixed strategy, while the first player always chooses a pure strategy best against the last strategy used by his opponent. That is to say, the second player always acts on the assumption that his opponent will continue to play with the frequency distribution of pure strategies which he has exhibited in the past, the first player on the assumption that his opponent will repeat his last play. For the classes of games concerned the iteration will repeat itself in cycles of predeterminate length, the end of each cycle being recognizable as a stage at which the first player's accumulated mixed strategy yields the same outcome for all strategies of the second player." (Author's summary)

J. Wolfowitz (Ithaca, N.Y.)

1238:

David, H. A. Tournaments and paired comparisons. Biometrika **46** (1959), no. 1/2, 139–149.

In the case of a knock-out tournament among $t=2^k$ players, the probability that player i' will beat player j' in the rth round ($r=1, \cdots, k-1$) is derived for given probabilities π_{ij} ($i, j, = 1, \cdots, t$) where π_{ij} is the probability that i will beat j, and that $\pi_{ij}+\pi_{ji}=1$.

A generating function is applied to enumerate the outcomes of Round Robin tournaments. Complete tables for $t<8$ players are given. This enables the tabulation of upper percentage points of x_1, the largest number of games won by any participant in a Round Robin tournament among t equally strong participants ($t=3, \cdots, 20$). Thus the statistic x_1 can be used to test the hypothesis that all players are equally strong.

Procedures are also given for testing that two specified players are equal, and that a particular player is of average strength. *L. E. Moses* (Stanford, Calif.)

1239:

★Mills, W. H. The four person game—finite solutions on the face of the cube. Contributions to the theory of games, vol. IV, pp. 125–143. Annals of Mathematics Studies, no. 40. Princeton University Press, Princeton, N.J., 1959. xi+453 pp. $6.00.

All finite solutions are determined for those zero-sum four-person games that correspond to the faces of the von Neumann cube. Only those points lying on a diagonal joining opposite vertices correspond to games with finite solutions (a new seven point one).

A. Charnes (Evanston, Ill.)

1240:

Vorob'ev, N. N. On coalitional games. Dokl. Akad. Nauk SSSR **124** (1959), 253–256. (Russian)

A coalitional game is an ordered quadruple

$$(*) \qquad \langle I, \mathfrak{K}, \{S_i\}_{i\in I}, \{H_{i,K}\}_{i\in K\in\mathfrak{K}} \rangle,$$

where I, a finite set, is interpreted as the set of players; $\mathfrak{K}$, a complex of subsets of I, is interpreted as the set of possible coalitions (thus a player can participate in more than one coalition); S_i, a finite set associated with i, is interpreted as the set of strategies available to i; and $H_{i,K}$, a real-valued function on the set $S_I=\prod_{i\in I} S_i$, is interpreted as the payoff function for the player i in his capacity as a member of the coalition K. An element of S_I is called an equilibrium situation if an increase in the payoff to a player can be prevented by an appropriate change in the strategies of his coalition partners.

A non-coalitional game (identical with J. Nash's non-co-operative game) is a special case of (*), with $\mathfrak{K}$ consisting of single points.

A mixed coalitional strategy and a mixed equilibrium situation for coalitional games are defined, and the concept of a regular complex used. If $\mathfrak{K}$ is regular then there exists a mixed equilibrium situation.

J. Marschak (Pittsburgh, Pa.)

BIOLOGY AND SOCIOLOGY

1241:

Leslie, P. H.; and Gower, J. C. The properties of a stochastic model for two competing species. Biometrika **45** (1958), 316–330.

The authors give the results of computer calculations of normal approximations to the stochastic model presented earlier [P. H. Leslie, Biometrika **45** (1958), 16–31; MR **19**, 1245]. The extremes of the model are Model I with constant birth-rates and Model II with constant death-rates. Using both models for a system with an unstable stationary state, estimates (each based on sixty runs) of the probability of extinction of a particular species are given for different initial population sizes. Using Model I for a system with a stable stationary state, the variances and covariance of the population sizes are estimated and compared with an approximate theoretical development.

M. A. Stone (Cambridge, England)

1242:

Cureton, Edward E. A note on factor analysis: arbitrary orthogonal transformations. Psychometrika **24** (1959), 169–174.

A modification of the Gram-Schmidt process yields an easily constructed orthogonal transformation matrix which may be used to rotate a centroid, principal axis, or maximum likelihood factor matrix in a manner such that one of the new axes has predetermined direction. The procedure is illustrated by rotating a centroid factor matrix into an abbreviated bifactor matrix, the general factor being defined as the centroid of a specified subgroup of reasoning tests. *Author's summary*

INFORMATION AND COMMUNICATION THEORY

See also 932, 1143.

1243:

Darlington, Sidney. Linear least-squares smoothing and prediction, with applications. Bell. System Tech. J. **37** (1958), 1221–1294.

The author treats several basic problems in prediction theory by a unified method involving a skilful use of frequency domain techniques. A representative problem is that of finite memory prediction in which (a) the signal, $s(t)$, and noise, $n(t)$, are sample functions of uncorrelated stationary processes with rational spectral density functions; (b) the sum $s(\xi)+n(\xi)$ is given for $t-T\leq\xi\leq t$, T being a positive constant; (c) the class of allowable predictors is that of all linear time-invariant finite memory systems; and (d) an optimal predictor is one which minimizes the expectation $E\{[g(t)-s(t+\alpha)]^2\}$, where $g(t)$ is the output of the predictor and α ($\alpha>0$) is the prediction time. Let $Y_M(p)$ be the transfer function of an optimal predictor. Expressing $Y_M(p)$ in the form $A(p)+B(p)e^{-\tau p}$, where $A(p)$ and $B(p)$ are rational functions of p, the author obtains a set of necessary and sufficient conditions involving the poles and zeros of $A(p)$ and $B(p)$ under which $Y_M(p)$ is optimal. Discrete time-parameter and non-stationary processes are treated in a similar fashion, although in the non-stationary case the difficulty of factoring the covariance function (i.e., representing the covariance

$$R(t_1, t_2) = E\{[s(t_1)+n(t_1)][s(t_2)+n(t_2)]\}$$

in the form $R(t_1, t_2)=\int_{-\infty}^{\infty} k(t_1, \tau)k(\tau, t_2)d\tau$, where $k(t_1, \tau)=0$ for $t_1>\tau$) limit applications of the method to simple problems. *L. A. Zadeh* (Berkeley, Calif.)

1244:

★Nedoma, Jiří. The capacity of a discrete channel. Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956, pp. 143–181. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957. 354 pp. Kčs 34.00.

The author examines the coding theorem for a certain type of stationary discrete channel with (possibly infinite) memory. The channels considered have the following property: for each channel input sequence x, the probability distribution of any n successive channel outputs $y_t, \cdots, y_{t+n-1}$ depends only on the components $x_t, \cdots, x_{t+n-1}$ of x. The author introduces loss ("weight") and risk notions, but treats in detail only the cases risk = (frequency of incorrect blocks, using a block code) and risk = (frequency of incorrect symbols after decoding). A source is transmissible through a given channel if, for each $\varepsilon>0$, (risk) $<\varepsilon$ for some (sufficiently long) block code. Channel capacity $\mathfrak{C}$ is the supremum of source entropy rates of transmissible sources; the two risk functions above give the same $\mathfrak{C}$. The author proves $\mathfrak{C}\leq\mathfrak{C}_E$, where $\mathfrak{C}_E$ is the usual ergodic capacity, and gives an example with inequality. {Reviewer's note: One expects that $\mathfrak{C}=\mathfrak{C}_E$ will hold if the channel is such that the joint input-output process is ergodic for any ergodic sources.}

S. P. Lloyd (Murray Hill, N.J.)

1245:

Wolter, Hans. Zum Grundtheorem der Informationstheorie, insbesondere in der Optik. Physica **24** (1958), 457–475.

This paper examines critically two basic theorems used in applications of information theory to optics, namely the "sampling theorem" of C. Shannon and the "expansion theorem" of D. Gabor. It is claimed that neither theorem is strictly and generally true and examples are given in support of this claim. Modifications of the theorems are discussed. *E. Wolf* (Rochester, N.Y.)

SERVOMECHANISMS AND CONTROL

See also 931.

1246:

★Oppelt, W. (Redakteur) Anwendung von Rechenmaschinen bei der Berechnung von Regelvorgängen. Vorträge gehalten bei einer Tagung des Fachausschusses Regelungsmathematik der Gesellschaft für angewandte Mathematik und Mechanik in Düsseldorf am 8. November 1957. R. Oldenbourg, Munich, 1958. 128 pp. DM 16.80.

Computational methods of treating several problems associated with regulators for mechanical or electrical systems are discussed by various authors. The computing aids used range from graphic constructions, through special analog computers to general-purpose digital computers.

"A simple graphical method for deconvolution" (J. B. Reswick) outlines a graphical procedure for finding the transfer function of a network, when the autocorrelation and cross correlation of its input and output are known.

The analog computers whose schematics and use are discussed includes a polynomial calculator for finding roots (A. Leonhard), simulators of various degrees of generality (R. Herschel, O. Follinger and G. Schneider, E. Buhler). The Buhler article is a particularly comprehensive treatment of the analog representation of a mechanical regulator with friction.

Von Witsenhausen discusses the use of a analog computer in optimising the transient behavior of a regulator.

The application of "small electronic calculators" to problems of regulators (E. Bucovics) includes the use of such computers as the IBM 705, and IBM 604. Mention is made of an analog computer by Telefunken, though no details of this latter are given.

The book closes with three discussions on the general utility of analog computers in the practical design of regulators.

All the articles except Reswick's are in German.

G. R. Stibitz (Cambridge, Vt.)

1247:

Kupriyanova, L. I. On stability of a nonlinear control system with a neutral object. Avtomatika i Telemehanika **20** (1959), 127–134. (Russian. English summary)

A study of the stability of a nonlinear control system governed by the equations

$$\begin{aligned} \dot{\psi} &= -n(t)\xi \\ \sigma &= A_0\psi + A_1\dot{\psi} + A_2\ddot{\psi} \\ \dot{\xi} &= -\rho\xi + f(\sigma) \end{aligned} \tag{1}$$

in which ψ is the regulated coordinate, ξ is the coordinate of the regulating organ, $f(\sigma)$ represents the characteristic of the servomotor, the A's and ρ are constants, and $n(t)$ is a time-varying coefficient defined for $0 \leq t \leq T$, T being a positive constant. The regulated object is "neutral" in the sense that its equation (first equation in (1)) does not involve terms in ψ and $\dot{\psi}$. The system is treated, in effect, as if it were a constant coefficient system for each fixed $t \in [0, T]$. *L. A. Zadeh* (Berkeley, Calif.)

1248:

Pfeiffer, Alfred. Zur Stabilität von Folgereglern mit nichtlinearer Beziehung zwischen Führungsgrösse und Regelgrösse. Z. Angew. Math. Mech. **38** (1958), 472–479. (English, French and Russian summaries)

The author investigates the stability of a follow-up system in which the output y is supposed to be a nonlinear function $f(x)$ of the input x. The system has a linear transfer function in the forward path, a linear transfer function in the feedback path and a mixing device for forming a nonlinear error signal, which assures the relation $y = f(x)$. With the assumption that the inverse function $x = \varphi(y) \neq 0$ and $d\varphi/dy \neq 0$ in the entire working region, a nonlinear error signal $U = C[x - \varphi(\bar{y})]/(d\varphi/d\bar{y})$ will guarantee uniform stability of the follow-up system in the entire working region. $\bar{y} = y_{II}(p) \cdot y$, when y_{II} is the linear transfer function in the feedback path. *I. Flügge-Lotz* (Stanford, Calif.)

1249:

Halanay, A. Soviet research in the problem of stability of automatic control systems. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **12** (1958), no. 3 (26), 5–14. (Romanian)

1250:

Korolev, L. N. Circuit-switching function of a searching arrangement operating in accordance with a given table. Dokl. Akad. Nauk SSSR **125** (1959), 482–484. (Russian)

HISTORY AND BIOGRAPHY

1251:

Heller, Siegfried. Die Entdeckung der stetigen Teilung durch die Pythagoreer. Abh. Deutsch. Akad. Wiss. Berlin. Kl. Math. Phys. Tech. **1958**, no. 6, 28 pp.

The discovery of irrationals is ascribed to the fifth century B.C. Hippasus of Metapontus, probably resulting from his efforts to construct the regular pentagon. It is conjectured that he or his followers also were led to a construction for the division of a segment in mean and extreme ratio by the consideration of a sequence of number pairs satisfying the relation

$$(d_n - s_n)d_n - s_n^2 = \begin{cases} 1, & n \text{ even}, \\ -1, & n \text{ odd}. \end{cases}$$

The conjecture is made plausible by the exhibition of an analogous sequence and equation which appears in the ancient sources and which arose in connection with the side and diagonal of the square. *E. S. Kennedy* (Beirut)

1252:

Stamatis, Evangelos. The theory of sets by Plato. Prakt. Akad. Athēnōn **33** (1958), 299–303. (Greek. English summary)

The article begins with Cantor's remark that his definition of a set as "eine Vielheit als Einheit gedacht" is closely related to the passage in the Platonic dialogue "Philebus" in which a set is described as an "idea" which is "mixed". It continues with two quotations from the "Parmenides" containing, e.g., the words "the part is a part, not of the many nor of all, but of a single . . . concept which we call the whole". Then follow four propositions from the commentator Proclus concerned chiefly with the tenet that a set exists before the existence of its elements, a tenet which is related to the Platonic theory of ideas in that the ideas were thought by Plato to possess a greater reality than their embodiments in the actual world. The discussion of Proclus, taken largely from his work "Theological Elementalism", has some mathematical interest as a forerunner of modern difficulties with the theory of sets. *S. H. Gould* (Providence, R.I.)

1253:

Szabó, Á. Wie ist die Mathematik zu einer deduktiven Wissenschaft geworden? Acta Antiqua Acad. Sci. Hungar. **4** (1956), 109–152. (Russian summary)

In attempting to answer the question posed in the title of the paper, the author first examines answers proposed by others. It has been suggested, for instance, that the notion of proof emerged from the sophists' dialectic developed in the Greek city state democracies. Another suggestion is that the Greeks, being confronted with different approximations to the same constant (π, say) used in Egyptian and Babylonian mathematics, thus realized the necessity of rigorous deductive methods. The author finds these and other notions useful, but inconclusive and faulty. He develops the idea that the logical concepts of the self-consistent proposition, the excluded middle, and the indirect method of proof, which were extensively applied by the Pythagoreans in mathematics, were not originated by them but by the Eleactic school of philosophy which flourished in southern Italy in the fifth and sixth centuries B.C. *E. S. Kennedy* (Beirut)

1254:

Szabó, Árpád. How mathematics became a deductive science. I, II. Mat. Lapok **8** (1957), 8–36, 232–247. (Hungarian)

Hungarian version of the paper reviewed above.

1255:

Stamatis, Evangelos. The pre-Socratic philosophers. Thales of Miletus, the great scholar and philosopher. Episteme kai Techne **116** (1959), 15 pp. (Greek)

A short history of Miletus and a biography of Thales, followed by a discussion of his works under the headings: geometry, astronomy, physics, philosophy, apophthegms. *S. H. Gould* (Providence, R.I.)

1256:

Dyck, Martin. Goethe's thought in the light of his pronouncements on applied and misapplied mathematics. PMLA **73** (1958), 505–515.

A discussion of several passages from Goethe's works, with the purpose of showing that his remarks on the

mystical and dialectic misuse of mathematics are to be regarded in the same light as his remarks on misuse of language in general. The widespread opinion that Goethe's attitude toward mathematics and mathematicians was unduly unfavorable is therefore shown to be an exaggerated one.

1257:

Vekua, I. N. The progress of Soviet mathematics. Pokroky Mat. Fys. Astr. **3** (1958), 402–409. (Czech.)

1258:

★**Dijksterhuis, E. J. Die Mechanisierung des Weltbildes.** Ins Deutsche übertragen von Helga Habicht. Springer-Verlag, Berlin–Göttingen–Heidelberg, 1956. vii + 594 pp. DM 36.00.

This book is a very complete survey of the development of Western science through the seventeenth century. It starts with the Greek philosophers, in particular, with the school of Pythagoras which had considerable influence on Plato and on the Neo-platonists. The corpuscular theory of the early Greeks as typified by Empedocles, Democritus, and Lucretius, is also traced. Quite naturally, the philosophies of Plato and of Aristotle receive a prominent place because of their great influence on Western thought, directly from the Greeks and indirectly through the Arabs. Considerable space is given to a discussion of mathematics, mathematical physics, astronomy, physics, chemistry, and astrology of these early times.

The author then proceeds to a consideration of the natural sciences in the Middle Ages, that is, through the fourteenth century, including the contributions of the Arabs. In a review of such a book as this, it is not possible to even indicate how the author traces the development of science through the work of many individuals. Suffice it to say that all the names familiar to most scientists are included as well as names of many whose works are not so familiar.

The period covered by the fifteenth, sixteenth and seventeenth centuries is the time of the Renaissance and the birth of classical science. This was a period of great activity in the natural sciences. During this time lived such well-known persons as Leonardo da Vinci, Copernicus, Kepler, Torricelli, Galileo, Huygens, Gilbert, Francis Bacon, Descartes, Boyle, and Pascal. Their works, as well as those of a host of lesser-known persons, are appraised by the author and their contributions to the rise of the classical period are explained. The book completes its survey with a discussion of the works of Isaac Newton.

In addition to a brief closing statement, the book contains several pages of notes referring to a rather extensive bibliography. The reviewer noted that there were no bibliographical references to translations, with or without text, of Arab authors whose contributions were mentioned. This can be remedied by obtaining such titles from the books by Mieli and Sarton listed in the bibliography.

This is a very interesting book. It should be of value to all scientists who have not had an opportunity to study the history of science but who should know how science developed from early beginnings. It reveals how the work of Newton was the culmination of the efforts of a long line of distinguished predecessors. *E. B. Allen* (Troy, N.Y.)

1259:

★**Шмидт, О. Ю. Избранные труды. Математика.** [**Šmidt, O. Yu. Selected works. Mathematics.**] Izdat. Akad. Nauk SSSR, 1959. 315 pp. (1 plate) 17.70 rubles.

This volume contains a photograph of Šmidt, a complete list of biographical dates (including the dates of publication of all his works in varied fields) and an almost complete selection of his mathematical works.

MISCELLANEOUS

1260:

★**Mitrinović, Dragoslav S. Zbornik matematičkih problema sa prilozima i numeričkim tablicama, I, II.** [**Collection of mathematical problems with applications and numerical tables. I. (2nd ed. revised and enlarged), II.**] Univerzitet u Beogradu, Belgrade, 1958. Vol. I, xxiv + 351 pp.; vol. II, viii + 334 pp. $16.00.

Volume 1: Introductory problems (175 problems); algebra (231), analytical geometry (214), differential and integral calculus (320), mathematical induction (100); problems from various fields (200); supplements (on 9 topics); a small atlas of curves; numerical tables.

Volume 2: Determinants and matrices (120); series (53); complex numbers and functions (247); differential equations (308); problems from various fields (69); supplements (on 6 topics); vector analysis (53), by D. Mihailović; differential geometry (161), by J. Ulčara; numerical tables.

These volumes comprise a collection of problems in pure mathematics useful for students in the Faculty of Electrical Engineering in the University of Belgrade. The answers are often given and there are occasional hints. The interest of the book is heightened by bibliographical remarks, mathematical aphorisms and the like.

S. H. Gould (Providence, R.I.)

1261:

★**Pedoe, Dan. The gentle art of mathematics.** With drawings by Griselda El Tayeb. The Macmillan Co., New York, 1959. 143 pp. $3.50.

A short, interesting book for the layman. There are 9 chapters: mathematical games, chance and choice, where does it end (i.e., transfinite numbers), automatic thinking (logic, algebra of classes, etc.), two-way stretch, rules of play (elementary algebra, groups, etc.), an accountant's nightmare (infinite series), double talk (antinomies), what is mathematics.

{Although the book was first copyrighted in 1958, the writing of it appears to have been complete in 1955, since the last chapter includes a description of Mathematical Reviews not applicable after that date.}

Fulton, C. M. ... 842
Gaffey, W. R. ... 948
Gaffney, M. P. ... 892
Gale, D. ... 914
Ganea, T. ... 874
Ganzburg, I. M. ... 788
Gasiorowicz, S. ... 1152
Gazis, D. C. ... 1039, 1040
Gel'man, I. V. ... 827
Gerstenhaber, M. ... 1226
Gheorghiță, Ș. L. ... 1118
Ghermanescu, M. ... 796
Ghosh, Ambarish ... 1080, 1081
Gillert, H. ... 1221, 1222
Gillis, J. ... 983
Girshick, M. A. ... 952
Gladyšev, E. G. ... 927
Glansdorff, P. ... 1099
Glasser, M. L. ... 1059
Glomot, C. ... 1087
Gluškov, V. M. ... 698, 699
Goldstein, S. = Goshen, S.
Goldstine, H. H. ... 965
Golomb, M. ... 962
Gomory, R. E. ... 1230
Gorenstein, D. ... 688
Gosar, P. ... 1131
Goshen, S. ... 1164, 1167
Gower, J. C. ... 1241
Greene, J. M. ... 1122
Greenspon, J. E. ... 1032
Greenwood, R. E. ... 968
Gremillard, J. ... 1206
Griffiths, H. B. ... 872
Grigor'eva, E. A. ... 764
Grinberg, È. Ya. ... 830
Groves, A. D. ... 1224
Guderley, K. G. ... 1101
Guillaume, M. ... 649, 654
Guy, R. ... 817
Haimovici, A. ... 768
Halasnay, A. ... 1249
Halbwachs, F. ... 1168, 1189
Hammer, P. C. ... 970
Hämmerlin, G. ... 947
Hamming, R. W. ... 973
Hanke, W. ... 1119
Harary, F. ... 653
Hart, W. L. ... 705
Haser, L. ... 1216
Hasimoto, H. ... 1078
Heins, M. ... 719
Heinz, E. ... 743
Heller, S. ... 1251
Henry, L. ... 1171
Henze, E. ... 928
Herrmann, G. ... 1056
Hetényi, M. ... 1014
Heuser, H. ... 815, 816
Hillion, P. ... 1188
Hirschfelder, J. O. ... 1108, 1109, 1110
Hlavatý, V. ... 899
Hodge, P. G., Jr. ... 1045
Hooper, J. W. ... 945
Horadam, A. F. ... 836
Horiuchi, F. Y. ... 1062
Hornyik, L. ... 988
Hosszú, M. ... 988
Hsu, C. S. ... 740
Hsu, L. C. ... 734, 966
Huet, D. ... 758
Hwang, Cheng-chung ... 893
Idlis, G. M. ... 1204
Infantozzi, C. A. ... 856
Infeld, L. ... 1193
Ingleton, A. W. ... 655
Isaev, P. S. ... 1153
Ishihara, S. ... 891
Ivović, V. A. ... 1033
Jäckel, H. ... 1139
Jain, M. K. ... 1115
Jaworowski, J. W. ... 859
Joga Rao, C. V. ... 1047
Johnson, K. L. ... 1008, 1009
Johnson, M. W. ... 1019
Johnson, N. L. ... 950
Jona-Lasinio, G. ... 1160
Jones, A. ... 880
Kaczmarz-Steinhaus, B. Z. ... 1123
Kahane, J. P. ... 794
Kajiura, K. ... 1074
Kajiwara, J. ... 732
Kakutani, S. ... 755
Kaliski, S. ... 1031
van Kampen, N. G. ... 1120
Kantorović, L. V. ... 808
Kaplan, S. A. ... 1210, 1211
Karapetyan, S. E. ... 880
Karlin, S. ... 923
Kärrholm, G. ... 1064
Karzel, H. ... 890
Katsoka, S. ... 1231
Kawai, N. ... 1111
Kay, I. ... 933
Kazanceva, G. E. ... 1034
Keller, J. B. ... 1070, 1071, 1130
Kelly, D. C. ... 1123
Keogh, F. R. ... 786
Kiesow, H. ... 915
Kimball, A. W. ... 939
Kirkwood, J. G. ... 999
Klahr, C. N. ... 1234
Kleman, M. ... 1190, 1191, 1192
Knaster, B. ... 860
Kobayashi, K. ... 835
Kogbetliantz, E. G. ... 964
Komar, A. ... 1196
Komatu, Y. ... 714
Kousis, P. ... 716
Kopal, Z. ... 959
Kopylov, G. N. ... 1103
Kornelčuk, N. P. ... 793
Korolev, L. N. ... 1250
Korolyuk, V. S. ... 981
Kotzig, A. ... 877
Kövari, T. ... 723
Krasinkar, R. H. ... 1002
Král, J. ... 711
Kranzer, H. C. ... 1071
Krasil'ščikova, E. A. ... 1068, 1069
Krasil'shchikova, E. A. = Krasil'ščikova, E. A.
Kreyszig, E. ... 724
Kruskal, M. D. ... 1122
v. Krzywoblocki, M. Z. ... 1062
Kukudžanov, S. N. ... 1036
Kukudžanov, V. N. ... 1044
Kundt, W. ... 823
Kupriyanova, L. I. ... 1247
Kurepa, S. ... 695, 708
Kuwabara, S. ... 1063
Kynch, G. J. ... 1079
Kyner, W. T. ... 748
Laasonen, P. ... 975
Ladyženskaya, O. A. ... 1085
Lagally, M. ... 884
Lagrange, R. ... 837
Lakshmana Rao, S. K. ... 1114
Lancaster, H. O. ... 944
Landau, L. D. ... 985, 1172
Leach, E. ... 939
Learner, A. ... 979
Lebedev, N. N. ... 1006
Lee, T. D. ... 1174
deLeeuw, K. ... 819, 820
Leipnik, R. ... 1143
Lelek, A. ... 860
Leslie, P. H. ... 1241
Levy, B. R. ... 1130
Liffšic, E. M. ... 985
Linder, B. ... 1109
Lindsay, J. H., Jr. ... 875
Linskii, V. S. ... 982
Lions, J. L. ... 757
Lipkin, H. J. ... 1166, 1167
Litvin-Sedoi, M. Z. ... 989
Ljubarskiĭ, G. L. ... 1117
Logunov, A. A. ... 1153, 1154, 1155
Lonwater, A. J. ... 862
Lombardi, L. ... 831
Lord, W. T. ... 1107
Loroñoz, M. ... 728
Lovemen, P. ... 945
Lucas, H. L. ... 949
Lyubčenko, G. G. ... 847
Madansky, A. ... 966
Majumdar, S. Datta. = Datta Majumdar, S.
Malinskiĭ, K. K. ... 829
Malyšev, A. V. ... 858
Mandelbrojt, S. ... 796
Mang, H. J. ... 1184
Maradudin, A. A. ... 1170
Marchuk, G. I. ... 1001
Marenkova, A. F. ... 1106
Margenau, H. ... 1123
Markus, L. ... 755
Marmion, A. ... 834
Marx, G. ... 1159
Masera, J. L. ... 756
Matthews, G. ... 813a.1
Matuzevichus, A. ... 870
McConnell, J. ... 1141
McGregor, J. ... 923
Medvedev, B. V. ... 1150
Meister, E. ... 1087
Mercer, A. ... 930
Merrill, J. A. ... 953
Mesner, D. M. ... 951
Meyer zur Capellen, W. ... 987
Michaels, J. E. ... 1062
Mikeladze, M. ... 1048
Mikeladze, Š. E. ... 971
Mikhailus, F. F. ... 1001
Mikodajska, Z. ... 747
Miles, J. W. ... 1106
Miller, J. B. ... 804
Mills, W. H. ... 1239
Milne, W. E. ... 976
Milne-Thomson, L. M. ... 1013
Mimura, Y. ... 1076
de Mira Fernandes, A. ... 885
Mișicu, M. ... 1012
Mitrinović, D. S. ... 1200
Molotkov, L. A. ... 1043
Molski, R. ... 858
Monin, A. S. ... 1096
Moór, A. ... 887
de Morsier, A. ... 995
Morikawa, H. ... 671, 672
Morozov, N. F. ... 1018
Morrison, D. ... 974
Morse, M. ... 799, 800
Motzkin, T. S. ... 705
Mrówka, S. ... 855
Mulholland, H. P. ... 722
Münch, G. ... 1093
Murgescu, V. ... 897, 898
Murray, F. J. ... 965
Naas, G. ... 1215
Nagami, K. ... 862
Nagao, H. ... 693
Nakamori, K. ... 678
Nakaoka, M. ... 864
Narayana, G. L. ... 1094
Nariai, H. ... 1213
Nedoma, J. ... 1244
von Neumann, J. ... 965
Niše, V. ... 839
Nijenhuis, A. ... 890
Ninomiya, N. ... 776
Nishijima, K. ... 1175
Nitš, M. M. ... 996
Nolley, J. W. ... 968
Novošilov, Yu. V. ... 1180
Noyes, H. P. ... 1152
Nozières, P. ... 1068
Nuttš, J. G. ... 1057
Obrechkoff, N. ... 707
Ogorodnikov, K. F. ... 1212
Ohji, M. ... 1095
Okabeyashi, T. ... 1162
Opial, Z. ... 751
Oppelt, W. ... 1246
d'Orgeval, B. ... 838
Osborn, H. ... 780
Otmahova, I. P. ... 1097
Owen, D. B. ... 954
Pai, S. I. ... 1116
Pais, A. ... 1104
Palamà, G. ... 661
Panzone, R. ... 809
Pao, Yoh-Han ... 1113
Paraeyuk, O. S. ... 1163
Parisot, G. ... 1232
Park, D. ... 1187
Parodi, M. ... 712
Pasynkov, B. ... 853
Patry, J. ... 741, 742
Pearl, M. ... 674
Pearson, C. E. ... 1003
Pearson, E. S. ... 937
Pedoe, D. ... 1251
Peters, J. ... 772
Peixoto, M. M. ... 753
Periman, H. S. ... 978
Perron, O. ... 720
Person, L. N. ... 997
Petraleo', G. L. ... 1042
Petrina, D. Ya. ... 1163
Petrov, L. M. ... 792
Pettineo, B. ... 704
Peyser, G. ... 725
Pfanzagl, J. ... 942
Pfeiffer, A. ... 1245
Pham, Mau Quan ... 1199
Picoard, S. ... 683, 684
Piechocki, W. ... 1065
Pierce, J. ... 1083
Pillai, K. C. Sreedharan ... 940, 946
Pines, D. ... 1064, 1124, 1135
Piranian, G. ... 812
Pitman, J. ... 670
Plotkin, B. I. ... 686
Polivanov, M. K. ... 1150
Polovin, R. V. ... 1117
Pompilj, G. ... 955
Popescu, I. L. ... 1086
Postnikov, A. G. ... 663, 664, 665, 666
Prodhan, Trilochan ... 1121
Prins, G. ... 653
Prohorov, L. V. ... 1180
Przeworska-Rolewicz, D. ... 797
Pyateckiĭ, I. I. ... 663, 664
Pyateckiĭ-Šapiro, I. I. ... 728
Quesnouille, M. H. ... 936
Rado, R. ... 652
Ragab, F. M. ... 733, 735
Raggi, B. ... 1132
Raimi, R. A. ... 810, 811
Rankin, R. A. ... 721
Rappoport, M. G. ... 964
Rasiowa, H. ... 650
Rayner, C. B. ... 1200
Regirer, S. A. ... 1082
Reid, R. O. ... 1073
Reinhart, B. L. ... 903
Reissner, E. ... 1019
Reuter, G. E. H. ... 918, 919
Reynolds, R. R. ... 976
Rimbushko, A. P. ... 1197, 1198
Rice, B. ... 1166
Rickstynya, V. Ž. ... 726
Riesel, H. ... 657
Riley, N. ... 1077
Rivlin, R. S. ... 669, 1022
Robin, L. ... 724
Robson, B. A. ... 978, 979
Rogers, C. A. ... 847
Rohlin, V. A. ... 865
Rohrlich, F. ... 1217, 1218
Rózsa, P. ... 1051
Rubinštein, G. Š. ... 806
Rukhadze, A. A. ... 1126
Rzewuski, J. ... 681, 1156
Saaty, T. L. ... 1253
Sade, A. ... 696
Salem, A. ... 1173
Salem, R. ... 791, 794
Salter, H. E. ... 960
Samson, P., Jr. ... 940
Samsonovič, S. ... 656
Sankaranarayanan, R. ... 1045
Sasao, S. ... 854
Scaglioni, L. ... 900
Scherbakov, R. N. ... 881

Schäffer, J. J. 680, 756, 801
Scheffler, H. 934
Scherk, P. 976b
Schild, A. 1236
Schoenberg, I. J. 961
Schreiber, S. 687
Schremp, E. J. 1178
Schubert, H. 992
Schumann, W. 1049
Schwerdtfeger, H. 676a
Šćigolev, B. M. 1207
Seki, T. 861
Sellars, J. R. 1089
Serber, R. 1154
Shen, S. F. 1140
Shibata, T. 1186
Shirokov, I. M. 1146
Shirokov, M. F. 1102
Sholander, M. 685
Siegel, C. L. 844
Sierpiński, W. 851
Silverman, R. A. 933
Simon, A. B. 697
Šmidt, O. Yu. 1259
Smith, A. M. O. 1063
Smith, C. S. 930
Smith, E. S. 1224
Smith, G. F. 689
Smith, R. B. 679
Sobol', I. M. 958
Sobolev, V. V. 1209
Söhngen, H. 1067
Sokolov, Yu. D. 977
Sokolowski, M. 1054
Solomon, L. 1015
Sommer, F. 730
Souriau, J.-M. 1202
Spence, R. D. 1149
Spencer, L. V. 1128
Speranza, F. 901
Squire, W. 1024
Stamsås, K. 1252, 1253
Stanuković, K. P. 1210
Stavroulakis, N. 894
Stevens, K. W. H. 1144
Stewartson, K. 1086
Stollow, S. 718
Stoll, W. 729
Stoller, L. 974
Stroud, A. H. 970
Strubecker, K. 878
Subba Rao, M. V. 667
Sumitomo, T. 911
Švarc, A. S. 865, 871
Szabó, Á. 1253, 1254
Szarski, J. 746
Székely, D. L. 646
Szlągowski, F. 1020
Tacunyak, P. I. 1163
Takács, L. 929
Takizawa, S. 867
Tamada, K. 1084
Tasi, G. 1051
Tavkhelidze, A. N. 1154, 1155
Teghem, J. 782
Teodorescu, P. P. 1025, 1026, 1027, 1028, 1029a–b
Ter-Mikaelyan, M. L. 1158
Terracini, A. 883
Thirring, W. E. 1157
Thom, R. 866, 910
Thoma, E. 821
Thüring, B. 980
Tihomirova, E. S. 904
Timan, A. F. 788
Ting, Lu 1100
Tingey, F. H. 962
Todd, J. 724
Togari, Y. 731
Tomilov, E. D. 1061
Toponogov, V. A. 850
Transue, W. 799, 800
Trustrum, G. B. 845
Tryon, R. C. 947
Tull, J. P. 659
Turpin, M. 759
Tutte, W. T. 840
Uchiyama, S. 669
Ueno, Y. 1213
Ufliand, I. S. 1006
Urbanik, K. 920a–b
Vakulenko, A. G. 1021
Vekua, I. N. 1257
Vigier, J.-P. 1188, 1189
Vincensini, P. 882
Vituškin, A. G. 787
Vladimirov, V. S. 958
Vlasov, A. A. 1205
Voda, J. J. 1062
Vodička, V. 1010
Volkov, Yu. A. 908
Volpato, M. 737, 761
Vopěnka, P. 863
Vorob'ev, N. N. 1240
Vrănceanu, G. 896
Waelbroeck, L. 824
Wagner, H. M. 1227, 1235
Wagner, R. 1075
Wagner, R. W. 780
Walsh, J. L. 725
Wang, Rou-hwai 754
Wang, Shou-jen 921
Watanabe, S. 912
Waterman, P. C. 1041
Wazewski, T. 746
Weiss, G. H. 1170
Wheelon, A. D. 1093
Whitin, T. M. 1235
Whitney, H. 889
Whittle, P. 925
Wiesen, J. M. 964
Wild, J. 677
Wild, W. 1184
Wilder, R. L. 873
Williams, M. L. 1008
Winogradzki, J. 694, 1185
Wirsing, E. 660
Witten, L. 1214
Wolter, H. 1245
Wu, Tsu-teh 869
van Wyk, C. B. 1142, 1177
Yakubovič, V. A. 749, 750
Yamada, H. 1072
Yamamoto, T. 991
Yang, C. N. 1174
Yih, Chia-Shun 1090
Young, L. C. 832a–b
Zadoyan, M. A. 1050
Zaguskin, V. L. 849
Zaidman, S. 803
Zandere, M. K. 703
Zaretskii, D. F. 1000
Zarovnyĭ, V. P. 833
Zassenhaus, H. 675
Zemanian, A. H. 1135
Zemmer, J. L. 682
Zeuli, T. 1007
Zierler, N. 781
Zinger, A. A. 941
Ziza, O. A. 785
Zwick, S. A. 1091, 1092
Zyrianov, P. S. 1165